Chapter 2
From the second unknown to the symbolic equation

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Abstract The symbolic equation slowly emerged during the course of the sixteenth century as a new mathematical concept as well as a mathematical object on which new operations were made possible. Where historians have often pointed at François Viète as the father of symbolic algebra, we would like to emphasize the foundations on which Viète could base his *logistica speciosa*. The period between Cardano's *Practica Arithmeticae* of 1539 and Gosselin's *De arte magna* of 1577 has been crucial in providing the necessary building blocks for the transformation of algebra from rules for problem solving to the study of equations. In this paper we argue that the so-called "second unknown" or the *Regula quantitates* steered the development of an adequate symbolism to deal with multiple unknowns and aggregates of equations. During this process the very concept of a symbolic equation emerged separate from previous notions of what we call "co-equal polynomials".

Key words: Symbolic equation, linear algebra, Cardano, Stifel, regula quantitates.

> L'histoire de la résolution des équations à plusieurs inconnues n'a pas encore donné lieu à un travail d'ensemble satisfaisant, qui donnerait d'ailleurs lieu à d'assez longues recherches. Il est intimement lié aux progrès des notations algébriques. J'ai appelé l'attention sur le problème de la resolution des equations simultanées, chaque fois que je l'ai rencontré, chez les auteurs de la fin du XVIe et du commencement du XVIIe siècle. (Bosmans, 1926, 150, footnote 16).

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2.1 Introduction

This footnote, together with many similar remarks by the Belgian historian Father Henri Bosmans (S.J.), initiated our interest in the role of the second unknown or *regula quantitates* on the development of symbolism during the sixteenth century.[1] Indeed, the importance of the use of multiple unknowns in the process leading to the concept of an equation cannot be overestimated. We have traced the use and the development of the second unknown in algebraic problem solving from early Arabic algebra and its introduction in Europe until its last appearance in Jesuit works on algebra during the late seventeenth century. The first important step in abbaco algebra can be attributed to the Florentine abbaco master Antonio de' Mazzinghi, who wrote an algebraic treatise around 1380 (Arrighi 1967). Luca Pacioli almost literally copied the solution method in his *Summa* of 1494, and Cardano used the second unknown both in his *Arithmetica* and the *Ars Magna*. A second thread of influence is to be distinguished through the *Triparty* by Chuquet and the printed works of de la Roche and Christoff Rudolff. The *Rule of Quantity* finally culminates in the full recognition of a system of linear equation by Buteo and Gosselin. The importance of the use of letters to represent several unknowns goes much further than the introduction of a useful system of notation. It contributed to the development of the modern concept of unknown and that of a symbolic equation. These developments formed the basis on which Viète could build his theory of equations.

It is impossible to treat this whole development within the scope of a single chapter. The use of the second unknown by Chuquet (1489) and de la Roche (1520) and its spread in early sixteenth-century Europe is already treated in Heeffer (2010a). Its reception and development on the Iberian peninsula has recently be studied by Romero (2010). In this paper we will concentrate on one specific aspect of the second unknown – the way it shaped the emergence of the symbolic equation.

2.2 Methodological considerations

As argued in Heeffer (2008), the correct characterization of the Arabic concept of an equation is the act of keeping related polynomials equal. Two of the three translators of al-Khwārizmī's algebra, Guglielmo de Lunis and Robert of Chester use the specific term *coaequare*. In the geometrical demonstration

[1] References to the second unknown are found in Bosmans (1925-6) on Stifel, Bosmans (1906) on Gosselin, Bosmans (1907) on Peletier, Bosmans (1908a) on Nunez and Bosmans (1926) on Girard.

of the fifth case, de Lunis proves the validity of the solution for the "equation" $x^2 + 21 = 10x$. The binomial $x^2 + 21$ is coequal with the monomial $10x$, as both are represented by the surface of a rectangle (Kaunzner, 1989, 60):

> Ponam censum tetragonum abgd, cuius radicem ab multiplicabo in 10 dragmas, quae sunt latus be, unde proveniat superficies ae; ex quo igitur 10 radices censui, una cum dragmis 21, coequantur.

Once two polynomials are connected because it is found that their arithmetical value is equal, or, in the case of the geometrical demonstration, because they have the same area, the continuation of the derivation requires them to be kept equal. Every operation that is performed on one of them should be followed by a corresponding operation to keep the coequal polynomial arithmetical equivalent. Instead of operating on equations, Arabic algebra and the abbaco tradition operate on the coequal polynomials, always keeping in mind their relation and arithmetical equivalence. Such a notion is intimately related with the *al-jabr* operation in early Arabic algebra. As is now generally acknowledged (Oaks and Alkhateeb, 2007; Heeffer 2008; Hoyrup 2010, note 7), the restoration operation should not be interpreted as adding a term to both sides of an equation, but as the repair of a deficiency in a polynomial. Once this polynomial is restored – and as a second step – the coequal polynomial should have the same term added.

At some point in the history of algebra, coequal polynomials will transform into symbolic equations. This transformation was facilitated by many small innovations and gradual changes in permissable operations. An analysis of this process therefore poses certain methodological difficulties. A concept as elusive as the symbolic equation, which before the sixteenth century did not exist in its current sense, and which gradually transformed into its present meaning, evades a full understanding if we only use our current symbolic language. To tackle the problem we present the original sources in a rather uncommon format, by tables. The purpose is to split up the historical text in segments which we consider as significant reasoning steps from our current perspective. Each of these steps is numbered. Next, a symbolic representation is given which conveys how the reasoning step would look like in symbolic algebra, not necessarily being a faithful translation of the original source. Finally, a meta-description is added to explain the reasoning and to verify its validity. So, we have two levels of description: the original text in the original language and notations, and a meta-level description which explains how the reasoning would be in symbolic algebra. Only by drawing the distinction, we will be able to discern and understand important conceptual transformations. Our central argument is that once the original text is directly translatable into the meta-description we are dealing with the modern concept of a symbolic equation.

2.3 The second unknown

Before discussing the examples, it is appropriate to emphasize the difference between the rhetorical unknown and unknowns used in modern transliterations. Firstly, the method of using a second unknown is an exception in algebraic practice before 1560. In general, algebraic problem solving before the seventeenth century uses a single unknown. This unknown is easily identified in Latin text by its name *res* (or sometimes *radix*), *cosa* in Italian and *coss* or *ding* in German. The unknown should be interpreted as a single hypothetical value used within the analytic method. Modern interpretations such as an indeterminate value or a variable, referring to eighteenth century notions of function and continuity, do not fit the historical context. In solving problems by means of algebra, abbacus masters often use the term 'quantity' or 'share' or 'value' apart from the *cosa*. The rhetoric of abacus algebra requires that the quantities given in the problem text are formulated in terms of the hypothetical unknown. The problem solving process typically starts with "suppose that the first value sought is one cosa". These values or unknown quantities cannot be considered algebraic unknowns by themselves. The solution depends on the expression of all unknown quantities in terms of the *cosa*. Once a value has been determined for the *cosa*, the unknown quantities can then easily be determined.

However, several authors, even in recent publications, confuse the unknown quantities of a problem, with algebraic unknowns. As a result, they consider the rhetorical unknown as an auxiliary one. For example, in his commentary on Leonardo of Pisa's *Flos*, Ettore Picutti (1983) consistently uses the unknowns x, y, z for the sought quantities and regards the *cosa* in the linear problems solved by Leonardo to be an auxiliary unknown. The "method of auxiliary variable" as a characterization by Barnabas Hughes (2001) for a problem-solving method by ben-Ezra also follows that interpretation. We believe this to be a misrepresentation of the original text and problem-solving method.

The more sophisticated problems sometimes require a division into subproblems or subsequent reasoning steps. These derived problems are also formulated using an unknown but one which is different from the unknown in the main problem. For example, in the anonymous manuscript 2263 of the Biblioteca Riccardiana in Florence (c. 1365; Simi, 1994), the author solves the classic problem of finding three numbers in geometric proportion given their sum and the sum of their squares. He first uses the middle term as unknown, arriving at the value of 3. Then the problem of finding the two extremes is treated as a new problem, for which he selects the lower extreme as unknown. We will not consider such cases as the use of two unknowns, but the use of a single one at two subsequent occasions. We have given some examples of what should not be comprehended as a second unknown, but let us turn to a positive definition. The best characterization of the use of several unknowns is operational. We will consider a problem solved by several unknowns if all of the following conditions apply in algebraic problem solving:

1. The reasoning process should involve more than one rhetorical unknown which is named or symbolized consistently throughout the text. One of the unknowns is usually the traditional cosa. The other can be named *quantità*, but can also be a name of an abstract entity representing a share or value of the problem.
2. The named entities should be used as unknowns in the sense that they are operated upon algebraically by arithmetical operators, by squaring or root extraction. If no operation is performed on the entity, it has no operational function as unknown in solving the problem
3. The determination of the value of the unknowns should lead to the solution or partial solution of the problem. In some cases the value of the second unknown is not determined but its elimination contributes to the solution of the problem. This will also be considered as an instance of multiple unknowns.
4. The entities should be used together at some point of the reasoning process and connected by operators or by a substitution step. If the unknowns are not connected in this way the problem is considered to be solved by a single unknown.

In all the examples discussed below, these four conditions apply.

2.4 Constructing the equation: Cardano and Stifel

2.4.1 Cardano introducing operation on equations

As far as we know from extant abbaco manuscripts Antonio de' Mazzinghi was the first to use the second unknown (Arrighi, 1967). Surprisingly, this was not for the solution of a linear problem but for a series of problems on three numbers in continuous proportion (or geometric progression, further GP). The same problems and the method of the second unknown are discussed by Pacioli in his *Summa*, without acknowledging de' Mazzinghi (Heeffer, 2010b). Before turning to Cardano's use of the second unknown, it is instructive to review his commentary on the way Pacioli treats these – and hence, Mazzinghi's – problems. In the *Questionibus Arithmeticis*, the problem is listed as number 28 (Cardano, 1539, f. DDiii^{v}). Not convinced of the usefullness of the second unknown, he shows little consideration for this novel solution as it uses too many unnecessary steps ("Frater autem Lucas posuit ean et soluit cum maga difficultate et pluribus operationibus superfluis"). He presents the problem (2.1) with $a = 25$ instead of 36, as used by Pacioli.

$$\begin{array}{l} \frac{x}{y} = \frac{y}{z} \\ x + y + z = a \\ \frac{a}{x} + \frac{a}{y} + \frac{a}{z} = x + y + z = xyz \end{array} \tag{2.1}$$

The solution is rather typical for Cardano's approach to problem solving. The path of the least effort is the reduction of the problem to a form in which theoretical principles apply. Using his previously formulated rule,[2]

$$\frac{a}{x} + \frac{a}{y} + \frac{a}{z} = x + y + z, y = \sqrt{a}$$

he immediately finds 5 for the mean term. As the product of the three, $xyz = y^3 = 125$, is also equal to the sum of the three, the sum of the two extremes is 120. Applying his rule for dividing a number a into two parts in continuous progression[3] with b as mean proportional

$$\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b^2},$$

he immediately arrives at

$$\left(60 + \sqrt{3575}, 5, 60 - \sqrt{3575}\right)$$

ita soluta est.

This approach is interesting from a rhetorical point of view. Abbaco treatises are primarily intended to show off the skills of the master, often involving the excessive use of irrationals while an example with integral values would have illustrated the demonstration with the same persuasion. These treatises are, with the exception of some preliminaries, limited to problem solving only. With Pacioli, some recurring themes are extracted from his sources and treated in separate sections. Cardano extends this evolution to a full body of theory, titled *De proprietatibus numerorum mirisicis*, including 136 articles (Cardano 1539, Chapter 42). The problem is easily solved because it is an application of two principles expounded in this chapter.

[2] Cardano 1539, Chapter 42, art. 91, f. Iii^v: "Omnium trium quantitatum continuae proportionalium ex quarum divisione alicuius numeri proventus congregati ipsarum aggregato aequari debeat, media illius numeri radix erit nam est eaedem necessarioeveniunt quantum aggregatum est idem ex supposito".

[3] Cardano 1539, Chapter 42, art. 116, f. Ivi^r: "Si sint duo numeri utpote 24 et 10 et velis dividere 24 in duas partes in quarum medio cadat 10 in continua proportionalitate, quadra dimidium maioris quod est 12 sit 144. Detrahe quadratum minoris quod est 100 remanet 44, cuius R addita ad 12 et diminuta faciet duos numeros inet quos 10 cadit in medio in contuna proportionalitate, et erunt 12 p R 44 et 10 et 12 m R 44."

Using such solution method, he completely ignores Pacioli's use of two unknowns for this problem. However Cardano adopts two unknowns for the solution of linear problems in the *Arithmetica Practicae* of 1539. Six years later he even dedicates two chapters of the *Ars Magna* (Cardano, 1545) to the use of the second unknown. The last problem he solved with two unknowns is again a division problem with numbers in continuous proportion.

Cardano used the second unknown first in chapter 51 in a linear problem (Opera Omnia, IV, 73-4). He does not use the name *regula quantitates* but *operandi per quantitatem surda*, showing the terminology of Pacioli. He uses *cosa* and *quantita* for the unknowns but will later shift to *positio* and *quantitates* in the *Ars Magna*.[4]

Let us look at problem 91 from the *Questionibus*, as this fragment embodies a conceptual breakthrough towards a symbolic algebra. The problem is a complex version of the classic problem of doubling other's money to make equal shares (Tropfke 1980, 647-8; Singmaster 2004, 7.H.4). In Cardano's problem, three men have different sums of money. The first has to give 10 plus one third of the rest to the second. The second has to give 7 plus one fourth of the rest to the third. The third had 5 to start with. The result should be so that the total is divided into the proportion 3 : 2 : 1 (Cardano 1539, Chap. 66, article 91, ff. GGviii^v – HHi^v):

> Tres ludeba*n*t irati rapverunt peccunias suas & alienas cu*m* aute*m* p*ro* amicu*m* quievissent primus dedit secu*n*do 10 p 1/3 residui. Secundus dedit tertio 7 p residui & tertio ia*m* remansera*n*t 5 nummi & primus habuit 1/2 secundus 1/3 tertius 1/6 quaerit*ur* summa omniu*m*, & qua*n*tu*m* habuit q*ui*libet.

The meta-description in symbolic form is as follows:

$$\begin{array}{l} a-10-\frac{1}{3}(a-10)=\frac{1}{2}(a+b+c) \\ b+10+\frac{1}{3}(a-10)-7-\frac{1}{4}\big(b+10+\frac{1}{3}(a-10)-7\big)=\frac{1}{3}(a+b+c) \\ c+\frac{1}{4}\big(b+10+\frac{1}{3}(a-10)-7\big)=\frac{1}{6}(a+b+c) \\ c=5 \end{array}$$

Cardano uses the first unknown for a and the second for b ("Pone q*uod* primus habuerit 1 co. secu*n*dus 1 quan."). He solves the problem, in the standard way, by constructing the polynomial expressions, corresponding with the procedure of exchanging the shares. Doing so he arrives at two expressions. The first one is

$$x = 21\frac{4}{7} + 3\frac{6}{7}y$$

("igit*ur* detrae 1/8 co. ex 5/12 co remanent 7/24 co. et hoc aequivalet 6 7/24 p. 1 1/8 quan. quare 7 co. aequivale*n*t 151 p. 27 quan. quare 1 co. aequalet

[4] The same problem is solved slightly different in the *Ars Magna* and is discussed below.

21 4/7 p. 3 6/7 quan."). This expression for x would allow us to arrive at a value for the second unknown. Instead, Cardano derives a second expression in x

$$x = 101\frac{4}{5} + 1\frac{4}{5}y$$

("et q*ui*a 5/12 co. aequivalent etiam 42 5/12 p. quan. igitur 5 co. aequivalebunt 509 p. 9 quan. quare 1 co. aequivalent 101 4/5 p. 1 4/5 quan."). As these two expression are equal he constructs an equation in the second unknown:

$$21\frac{4}{7} + 3\frac{6}{7}y = 101\frac{4}{5} + 1\frac{4}{5}y$$

("igitur cu*m* etia*m* aequivaleat 21 4/7 p. 3 6/7 quan. erunt 21 4/7 p. 3 6/7 quan. aequalia 101 4/5 p. 1 4/5 quan. "). The text continues with: "Therefore, subtracting the second unknowns from each other and the numbers from each other this leads to a value of 39 for the second unknown. And this is the share of the second one." ("igit*ur* ta*n*de*m* detrahe*n*do quan. ex quan. et numeru*m* ex numero fiet valor qua*n*titatis 39 et t*antu*m habuit secu*n*dus"). However, the added illustration shows us something very interesting (see Figure 2.1).

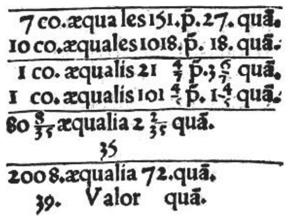
7 co. æquales 151. p̄. 27. quã.
10 co. æquales 1018. p̄. 18. quã.
1 co. æqualis 21 4/7 p̄. 3 6/7 quã.
1 co. æqualis 101 4/5 p̄. 1 4/5 quã.
80 8/35 æqualia 2 2/35 quã.
35
2008. æqualia 72. quã.
39. Valor quã.

Fig. 2.1: Cardano's construction of equations from (Cardano, 1539, f. 91^r)

The illustration is remarkable in several ways. Firstly, it shows equations where other illustrations or marginal notes by Cardano and previous authors only show polynomial expressions. As far as I know, this is the first unambiguous occurrence of an equation in print. This important fact seems to have gone completely unnoticed. Secondly, and supporting the previous claim, the illustration shows for the first time in history an operation on an equation. Cardano here multiplies the equation

$$80\frac{8}{35} = 2\frac{2}{35}y$$

by 35 to arrive at

$$2808 = 72y$$

The last line gives $39 = y$ and not 'y equals 39' which designates the implicit division of the previous equation by 72. The illustration appears both in the 1539 edition and the *Opera Omnia* (with the same misprint for 2808). As we discussed before, the term 'equation' should be used with caution in the context of early sixteenth-century practices. This case however, constitutes *the construction of an equation* in the historical as well as the conceptual sense. We have previously used an operational definition for the second unknown. Similarly, operations on an equation, as witnessed in this problem, support an operational definition of an equation. We can consider an equation, in this historical context, as a mathematical entity because it is directly operated upon by multiplication and division operators.

2.4.2 Michael Stifel introducing multiple unknowns

As a university professor in mathematics, Stifel marks a change in the typical profile of abbaco masters writing on algebra. In that respect, Cardano was a transitional figure. Cardano was taught mathematics by his father Fazio "who was well acquainted with the works of Euclid" (Cardano, 2002, 8). Although he was teaching mathematics in Milan, his professorship from 1543 was in medicine. His choice of subjects and problems fit very well within the abacus tradition. However, he did change from the vernacular of the abbaco masters to the Latin used for university textbooks. Stifel is more part of the university tradition studying Boethius and Euclid, but believed that the new art of algebra should be an integral part of arithmetic. That is why his *Complete Arithmetic* includes a large part on algebra (Stifel, 1544). Most of his problems and discussions on the cossic numbers, as he calls algebra, refer to Cardano. He concludes his systematic introduction with the chapter *De secundis radicibus*, devoted to the second unknown (ff. $251^v - 255^v$).

Several authors seem to have overlooked Cardano's use of the second unknown in the *Practica Arithmeticae*. Bosmans (1906, 66) refers to the ninth chapter of the *Ars Magna* as the source of Stifel's reference, but this must be wrong as the foreword of the *Arithmetica Integra* is dated 1543 and the *Ars Magna* was published in 1545. In fact, the influence might be in the reverse direction. Cifoletti (1993, 108) writes that "reading Stifel one wonders why the German author is so certain of having found most of his matter on the second unknown precisely in Cardano, i.e. in the *Practica Arithmeticae*. For, the *Ars Magna* would be more explicit on this topic". She gives the example of the *regula de medio* treated in chapter 51 of the *Practica Arithmeticae* (Opera, 87) and more extensively in the *Ars Magna* (Witmer, 92). She writes: "In fact, the rule Cardano gives for this case is not quite a rule for using several unknowns, but rather a special case, arising as a way to solve problems by 'iteration' of the process of assigning the unknown". However, Stifel's application of the *secundis radicibus* to linear problems unveils that he drew his inspiration from the problems in Cardano's *Questionibus* of Chapter 66, as the one discussed above. He makes no effort to conceal that:[5]

> Christoff Rudolff and Cardano treat the second unknown using the term *quantitatis*, and therefore they designate it as $1q$. This is at greater length discussed by Cardano. While Christoff Rudolff does not mention the relation of the second [unknown] with the first. On the other hand, Cardano made us acquainted with it by beautiful examples, so that I could learn them with ease.

Graciously acknowledging his sources, he adds an important innovation for the notation of the second an other unknowns. Keeping the cossic symbol ꝛ for the first unknown, the second is represented as $1A$, the third by $1B$, and so on, which he explains, is a shorthand notation for $1A$ꝛ and $1B$ꝛ, the square of $1A$ꝛ being $1A$ʒ. The use of the letters A, B and C in linear problems is common in German cossist manuscripts since the fifteenth century.[6] Although these letters are not used as unknowns, the phrasing comes very close to the full notation given by Stifel. For example, Widman writes as follows: "Do as follows, pose that C has $1x$, therefore having A 2ꝛ, because he has double of C, and B 3ꝛ, because he has triple".[7] Using Stifel's symbolism this would read as $1x$, $2Ax$ and $3Bx$. Although conceptually very different, the notation is practically the same. The familiarity with such use of letters made it an obvious choice for Stifel. Later, in his commentary on the *Coss* from Rudolff, he writes on Rudolff's use of 1ꝛ and $1q$., "However, I prefer to use $1A$ for $1q$.

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[5] Stifel (1544) f. 252^r: "Christophorus et Hieronymus Cardanus tractant radices secundas sub vocabulo Quantitatis ideo eas sic signant 1 q. Latius vero eas tractavit Cardanus. Christophorus enim nihil habet de commissionibus radicum sedundarum cum primis. Eas autem Cardanus pulchris exemplis notificavit, ita ut ipsas facile didicerim", (translation AH). In the edition of Rudolff's *Coss*, he adds: "Bye dem 188 exempl lehret Christoff die Regul Quantitatis aber auss vil oben gehandelten exemplen tanstu yetzt schon wissen wie das es teyn sonderliche regel sey... Das aber Christoff und auch Cardanus in sollichen fal setzen 1 q. Das ist 1 quantitet. Daher sie diser sach den nahmen haben gegeben und nennens Regulam Quantitatis" (Stifel 1553, 307).

[6] For example, the marginal notes of the C80 manuscript written by Johannes Widman in 1481, give the following problem (C80 f. 359^r, Wappler 1899, 549): "Item sunt tres socij, scilicet A, B, C, quorum quilibet certam pecuniarum habet summam. Dicit C: A quidem duplo plus habet quam ego, B vero triplum est ad me, et cum quilibet eorum partem abiecerit, puta A 2 et B 3, et residuum vnius si ductum fuerit in residuum alterius, proveniunt 24. Queritur ergo, quod quilibet eorum habuit, scilicet A et B, et quot ego". Høyrup (2010) describes an even earlier example by Magister Wolack of 1467, note 90.

[7] Ibid.: "Fac sic et pone, quod C habet 1 x, habebit ergo A 2x, quia duplum ad C, et B 3x, quia triplum".

because sometimes we have examples with three (or more) numbers. I then use 1𝔵, $1A$, $1B$, etc.”.[8]

Distinguishing between a second and third unknown is a major step forward from Chuquet and de la Roche who used one and the same symbol for both.[9] Before Stifel, there has always been an ambiguity in the meaning of the ‘second’ unknown. From now on, the second and the third unknown can be used together as in yz, which becomes $1AB$. However, Stifel’s notation system is not free from ambiguities. For the square of A, he uses $1A$ℨ, while ℨB should be read as the product of x^2 and B. The product of $2x^3$ and $4y^2$, an example given by Stifel, becomes 8𝔠Aℨ. A potential problem of ambiguity arises when we multiply $3x^2$ and $4z$, also given as an example. This leads to 12ℨB and thus it becomes very confusing that $12z^2x$ being the product of $12z^2$ and x is written as 12ℨB while $12z^2$ would be $12B$ℨ. Given the commutativity of multiplying cossic terms, both expressions should designate the same. The problem becomes especially manifest when multiplying more than two terms together using the extended notation. Stifel seems not be aware of the problem at the time of writing the *Arithmetica integra*.

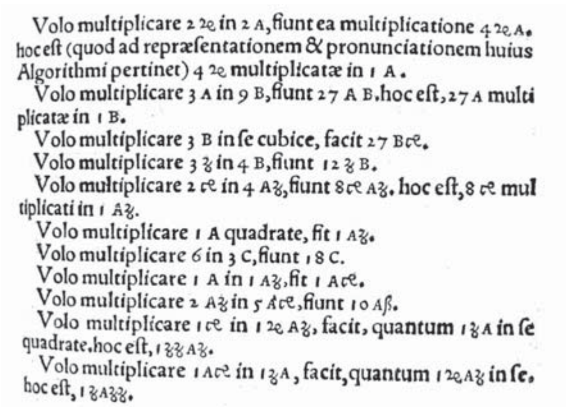
Volo multiplicare 2 𝔵 in 2 A, fiunt ea multiplicatione 4 𝔵 A. hoc est (quod ad repræsentationem & pronunciationem huius Algorithmi pertinet) 4 𝔵 multiplicatæ in 1 A.
Volo multiplicare 3 A in 9 B, fiunt 27 A B. hoc est, 27 A multiplicatæ in 1 B.
Volo multiplicare 3 B in se cubice, facit 27 B𝔠.
Volo multiplicare 3 ℨ in 4 B, fiunt 12 ℨ B.
Volo multiplicare 2 𝔠 in 4 Aℨ, fiunt 8 𝔠 Aℨ. hoc est, 8 𝔠 multiplicati in 1 Aℨ.
Volo multiplicare 1 A quadrate, fit 1 Aℨ.
Volo multiplicare 6 in 3 C, fiunt 18 C.
Volo multiplicare 1 A in 1 Aℨ, fit 1 A𝔠.
Volo multiplicare 2 Aℨ in 5 A𝔠, fiunt 10 Aß.
Volo multiplicare 1 𝔠 in 1 𝔵 Aℨ, facit, quantum 1 ℨ A in se quadrate. hoc est, 1 ℨℨ Aℨ.
Volo multiplicare 1 A𝔠 in 1 ℨ A, facit, quantum 1 𝔵 Aℨ in se. hoc est, 1 ℨ Aℨℨ.

Fig. 2.2: The rules for multiplying terms from Stifel (1545, f. 252^r)

The chapter on the *secundis radicibus* concludes with some examples of problems. Other problems, solved by several unknowns are given in *de exemplis* of the following chapters. Here we find solutions to many problems taken from Christoff Rudolff, Adam Ries and Cardano, usually including the correct ref-

[8] Stifel, 1553, f. 186^r: “Ich pfleg aber für 1q zusetzen 1A auss der ursach das zu zeyten ein exemplum wol drey (oder mehr) zalen fürgibt zu finden. Da setze ich sye also 1x, 1A, 1B etc”.
[9] For an extensive discussion of the second unknown in Chuquet, de la Roche and Rudolff and their interdependence see Heeffer (2010a).

erence. In the original sources, these problems are not necessarily treated algebraically, or by a second unknown. Let us look at one problem which he attributes to Adam Ries:[10]

> Three are in company, of which the first tells the second: if you give me half of your share, I have 100 fl. The second tells the third: if you give me one third of your share, then I have 100 fl. And the third tells the first: if you give me your sum divided by four, I have 100 fl. The question is how much each has.

The problem is slightly different from the example discussed above, in that the shares refer to the next one in the cycle and not to the sum of the others. The direct source of Stifel appears to be the unpublished manuscript *Die Coss* by Adam Riese, dated 1524 (Berlet 1860, 19-20). The problem is treated twice by Riese (problem 31, and repeated as problem 120). Although he uses the letters a, b and c, the problem is solved with a single unknown. Riese in turn might have learned about the problem from Fredericus Amann, who treated the problem in a manuscript of 1461, with the same values (*Cod. Lat. Monacensis* 14908, $155^r - 155^v$; transcription by Curtze, 1895, 70-1).
Stifel's version in modern notation is as follows:

$$a + \frac{b}{2} = 100$$
$$b + \frac{c}{3} = 100$$
$$c + \frac{a}{4} = 100$$

The solution is shown in Table 2.1. As a pedagogue, Stifel takes more steps than Cardano or the abacus masters before him. Line 8 is a misprint. Probably, the intention was to bring the polynomial to the same denominator as is done in step 13. This ostensibly redundant step shows the arithmetical foundation of the performed operations. Our meta-description gives the multiplication of equation (12) by 4 which makes line (13) superfluous. Stifel however, treats the polynomials as cossic numbers which he brings to the same denominator. Ten years later he will omit such operations as he acts directly on equations. The solution method is structurally not different from the one used by previous authors for similar linear problems. Note that Stifel does not use the second and third unknown in the same expression. The problem could as well be solved by two unknowns in which the second unknown is reused as by de la Roche. However, the fact that more than two unknowns are used opens up new possibilities and solution methods. How simply it may seem to the modern eye, the extension of the second unknown to multiple unknowns by Stifel was an important conceptual innovation.

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[10] Stifel 1553, f. 296r: "Exemplum quartum capitis huius, et est Adami. Tres sunt socij, quorum primus dicit ad secundum, Si mihi dares dimidium summae tua, tunc haberem 100 fl. Et secundus dicit ad tertium: Si mihi dares summae tuae partem tertiam, tunc haberem 100 flo. Et tertius ad primum dicit: Si tu mihi dares summae tuae partem quartam, tunc haberem 100 fl. Quaestio est, quantum quisque eorum habeat".

	Symbolic	**Meta description**	**Original text**
1	$a + \frac{b}{2} = 100$	premise	Quod autem primus petit â secundo dimidium summae, quam ipse secundus habet, ut ipse primus habeat 100 fl.,
2	$x + \frac{y}{2} = 100$	choice of first and second unknown	fatis mihi indicat, aequationemen esse inter 1x + 1/2A et 100 florenos. Sic aût soleo ponere fracta huiusmodi (1x + 1A)/2 aequatae 100 fl.
3	$2x + y = 200$	multiply (2) by 2	Ergo 2x + 1A aequantur 200 fl.
4	$y = 200 - 2x$	subtract 2x from (3)	Et 1A aequa*ntur* 200 fl – 2x. Facit ergo 1A, 200fl. – 2x id quod mihi reservo loco unius A. Habuit igitur primus 1x florenorum. Et secundus 200 fl. – 2x.
5	$z = c$	choice of third unknown	Et tertius 1B flor.
6	$y + \frac{z}{3} = 100$	premise	Petit autem secundus tertiam partem summae terti socij, ut sicispe secu*n*dus habeat 100 fl.
7	$200 - 2x + \frac{z}{3} = 100$	substitute (4) in (6)	Itaq*ue* iam 200 fl. – 2x fl + 1/3 B, aequantur 100 florenis.
8	$600 - \frac{6}{3}x + z = 100$	illegal	Sic ego soleo ponere huiusmodi fractiones, ut denominator respiciat totum numeratorem. Ut 600 – 6/3 x + B aequata 100.
9	$600 - 6x + z = 300$	multiply (7) by 3	Aequantur itaq*ue* 600 – 6x + B cum 300.
10	$z = 6x - 300$	add $6x + 600$ to (9)	Atq*ue* hac aequatione vides fatis, ut 1B resolvatur in 6x – 300. Et sic primus habuit 1x florenorum. Secundus 200 fl – 2x. Tertius 6x – 300.
11	$z + \frac{x}{4} = 100$	premise	Petit autem tertius partem quartam summae, quam habet primus, ut sic ipse tertius etiam habeat centum florenos.
12	$6\frac{1}{4}x - 300 = 100$	substitute (10) in (11)	Itaq*ue* 6 x – 300 aequant*ur* 100.
13	$\frac{25x-1200}{4} = 100$	from (12)	Item (25x – 1200)/4 aequant*ur* 100 fl.
14	$25x - 1200 = 400$	multiply (12) by 4	Et sic 25x – 1200 aequantur 400.
15	$25x = 1600$	add 1200 to (12)	Item 25x aequa*n*tur 1600 fl.
16	$x = 64$	divide (13) by 25	Facit 1x 64 fl.
17	$y = 200 - 128$	substitute (16) in (4)	Habuit igitur primus 1x, id est, 64 fl. Secundus habuit 200 – 2x.
18	$y = 72$	from (15)	i. 72 fl.
19	$z = 384 - 300$	substitute (18) in (10)	Et tertius habuit 6x – 300,
20	$z = 84$	from (19)	hoc est 84 fl.

Table 2.1: Stifel's exposition of the second unknown.

2.5 Cardano revisted: The first operation on two equations.

Cardano envisaged an *Opus perfectum* covering the whole of mathematics in fourteen volumes, published in stages (Cardano 1554). Soon after the publication of the *Practica arithmeticae*, he started working on the *Ars Magna*, which was to become the tenth volume in the series.[11] It was published by Johann Petreius in Nürnberg in 1545, who printed Stifel's *Arithmetica Integra* the year before as well as several other books by Cardano. We know that Cardano has seen this work and it would be interesting to determine the influence of Stifel.[12] The *Ars Magna* shows an evolution from the *Practica Arithmeticae* in several aspects. Three points are relevant for our story of the second unknown. Having learned that Tartaglia arrived at a solution to the cubic by geometrical reasoning, Cardano puts much more effort than before in delivering geometrical proofs, and this not only for the cubic equation. He also tries to be more systematical in his approach by listing all possible primitive and derivative cases of rules (which we call equations), and then by treating them separately. One of these primitive cases deals with two unknowns which he discusses in two chapters. Chapter IX is on *De secunda incognita quantitate non multiplicata* or the use of the second unknown for linear problems. Rules for solving quadratic cases are treated in Chapter X. Let us look at the first linear problem:[13]

> Three men had some money. The first man with half the other' would have had 32 *aurei*; the second with one-third the other', 28 *aurei*; and the third with one-fourth the others', 31 *aurei*. How much has each?

In modern notation the problem would be:

$$\begin{aligned} a + \tfrac{1}{2}(b+c) &= 32 \\ b + \tfrac{1}{3}(a+c) &= 28 \\ c + \tfrac{1}{4}(a+b) &= 31 \end{aligned} \qquad (2.2)$$

In solving the problem Cardano introduces the two unknowns for the share of the first and the second person ("Statuemus primo rem ignotam primam,

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[11] The dating can be deduced from the closing sentence of the Ars Magna: "Written in five years, may it last as many thousands" from Witmer (1968, 261).

[12] Cardano mentions in his biography that he is cited by Stifel in what must be the first citation index (2002, 220).

[13] Translation from Witmer (1968, 71). Witmer conscientiously uses p and q for *positio* and *quatitates* which preserves the contextual meaning. Unfortunately he leaves out most of the tables added by Cardano for clarifying the text, and replaces some of the sentences by formulas. As the illustrations and precise wording are essential for our discussion, I will use the Latin text from the *Opera Omnia* when necessary, correcting several misprints in the numerical values.

	Symbolic	Meta description	Original text
1	$a = x$	choice of first unknown	Statuemus primo rem ignotam primam,
2	$b = y$	choice of second unknown	secundo secundam rem ignotam
3	$c = 31 - \frac{1}{4}(x + y)$	substituting (1) and (2) in (2.2)c	tertio igitur 31 aurei, minus quarta parte rei, ac quarta parte quantitatis relicti sunt
4	$a + \frac{1}{2}(b + c) = 32$	premise	iam igitur vide, quantum habet primus, equidem si illi dimididium secundi et terti addicias, habiturus est aureos 32.
5	$a = 32 - \frac{1}{2}y - 15\frac{1}{2} + \frac{1}{8}x + \frac{1}{8}y$	substitute (2) and (3) in (4)	habet igitur per se aureos 32 m. 1/2 quan. m. 15 1/2 p. 1/8 positionis p. 1/8 quant.
6	$a = 16\frac{1}{2} - \frac{3}{8}y + \frac{1}{8}x$	from (5)	quare habebit 16 m. 3/8 quantitatis p. 1/8 pos.
7	$x = 16\frac{1}{2} - \frac{3}{8}y + \frac{1}{8}x$	substitute (1) in (6)	hoc autem sit aequale uni positioni
8	$\frac{7}{8}x + \frac{3}{8}y = 16\frac{1}{2}$	from (7)	erit 7/8 pos. et 3/8 quant. aequale 16 1/2
9	$7x + 3y = 132$	multiply (8) with 8	quare deducendo ad integra 7 pos. et 3 quant. aequabuntur 132.
10	$b + \frac{1}{3}(a + c) = 28$	premise	Rursus videamus, quantum habeat secundus, habet hic 28 si ei tertia pars primi ac tertij addatur
11	$\frac{1}{3}(a + c) = \frac{1}{3}x + 10\frac{1}{3} - \frac{1}{12}x - \frac{1}{12}y$	from (3) and (6)	ea est 1/3 pos. p. 10 2/3 m. 1/12 pos. m. 1/12 quant.
12	$\frac{1}{3}(a + c) = \frac{1}{4}x + 10\frac{1}{3} - \frac{1}{12}y$	from (11)	hoc est igitur pos. p. 10 1/3 m. 1/12 quant.
13	$b = 17\frac{2}{3} + \frac{1}{12}y - \frac{1}{4}x$	substitute (12) in (11)	abbice ex 28 relinquitur 17 2/3 p. 1/12 quant. m. pos. et tantum habet secundus.
14	$y = 17\frac{2}{3} + \frac{1}{12}y - \frac{1}{4}x$	substitute (2) in (14)	suppositum est autem habere illum quantitatem, quantitas igitur secunda, aequivalet 1/12 suimet, et 17 2/3 p. m. pos.

secundo secundam rem ignotam") (*Opera* III, 241). In the rest of the book the two unknowns are called *positio* and *quantitates*, abreviated as *pos.* and *quan.* They appear regularly throughout the later chapters, and in some cases Cardano uses *pos.* for problems solved with a single unknown.

Note how strictly Cardano switches between the role of two unknowns and the share of the first and second person by making the substitution steps of lines (7) and (14) explicit.

15	$\frac{11}{12}y + \frac{1}{4}x = 17\frac{2}{3}$	subtract $\frac{1}{12}y$ from (14) and add $\frac{1}{4}x$	abiectis communiter 1/12 quantitatis, et restituto m. alteri parti, sient 11/12 quan. p. pos aequalia 17 2/3,
16	$11y + 3x = 212$	multiply (15) by 12	quare 11 quant. p. 3 pos. aequalia erunt 212 multiplicatis partibus omnibus per 12 denominatorem.

The next part in the solution is the most significant with respect to the emerging concept of a symbolic equation. Historians have given a lot of attention to the *Ars magna* for the first published solution to the cubic equation, while this mostly is a technical achievement. We believe Cardano's work is equally important for its conceptual innovations such as the one discussed here.

The first occurrence of the second unknown for a linear problem is by an anonymous fifteenth-century abbaco master, author of Fond. prin. V.152.[14] The problem about four men buying an ox is by means of the second unknown reduced to two "linear equations", $7y = 13x + 4$ and $4y = 2x + 167$. Expressed in symbolic algebra it is obvious to us that by multiplying the two equations with the coefficients of y, we can eliminate the second unknown which leads to a direct solution. However, the author was not ready to do that, because he did not conceive the structures as equations. They are subsequently solved by the standard tool at that time, the rule of double false position. Cardano here marks a turning point in this respect. Having arrived at two equations in two unknowns Cardano gives a general method: [15]

> Now raise whichever of these you like to equality with the other with respect to the number of either x or y "(in positionum aut quantitatum numero"). Thus you may decide that you wish, by some method, that in $3x + 11y = 212$, there should be $7x$. Then, by using the rule of three, there will be
>
> $$7x + 25\frac{2}{3}y = 494\frac{2}{3}.$$
>
> You will therefore have, as you see,
>
> $$7x + 3y = 132 \text{ and } 7x + 25\frac{2}{3}y = 494\frac{2}{3}$$
>
> Hence, since $7x$ is the same in both, in both the difference between the quantities of y, namely 22 2/3, will equal the difference between the numbers, which is 362 2/3.

[14] Franci and Pancanti, 1988, 144, ms. f. 177^r: "che tra tutti e tre gli uomeni avevano 3 oche meno 2 chose e sopra a questo agiugnerò l'ocha la quale si vole chonperare, chos aremo che tra tutti e tre gli uomeni e l'ocha saranno 4 oche meno 2 chose, dove detto fu nella quistione che tra danari ch'anno tutti e tre gli uomeni e 'l chosto del'ocha erano 176. Adunque, posiamo dire che lle 4 oche meno 2 chose si vagliano 176, chosì ài due aguagliamenti". In Heeffer (2010b) it is argued that this text is by Antonio de' Mazzinghi or based on a text by his hand.

[15] Cardano 1663, *Opera* IV, 241. I have adapted Witmer's translation to avoid the use of the terms coefficient and equation, not used by Cardano (Witmer 1968, 72).

> Divide therefore, as in the simple unknown, according to the third chapter, 362 2/3 by 22 2/3; 16 results as the value of y and this is the second.

Using modern terms, this comes down to the following: given two linear equations in two unknowns, you can eliminate any of the unknowns by making their coefficients equal and adapting the other values in the equation. The difference between the coefficient of the remaining unknown will be equal to the difference of the numbers. Although the result is the same, the text does not phrase the procedure as a subtraction of equations. However, the table added by Cardano, which is omitted in Witmer's translation, tells a different story:

$$\begin{array}{c} 7x + 3y = 132 \\ 7x + 25\frac{2}{3}y = 494\frac{2}{3} \\ \hline 22\frac{2}{3}y = 362\frac{2}{3} \end{array}$$

The table shows a horizontal line which designates a derivation: "from the first and the second, you may conclude the third". This table goes well beyond the description of the text and thus reads: "the first expression subtracted from the second results in the third". He previously used the same representation for the subtraction of two polynomials, also subtracting the upper line from the lower one (Cardano 1663, IV, 20). Cardano never describes the explicit subtraction of two equations in the text. Even if he did not intend to represent it that way, his peers studying the *Ars magna* will most aptly have read it as an operation on equations. As such, this is the first occurrence of an operation involving two equations, a very important step into the development of simultaneous equations and the very concept of an equation.

A second point of interest for the story of the second unknown is an addition in a later edition of the *Ars Magna* (Cardano, 1570; 1663; Witmer p. 75 note 13). Cardano added the problem of finding three so that the following conditions hold (in modern notation):[16]

$$\begin{aligned} a + b &= 1\tfrac{1}{2}(a + c) \\ a + c &= 1\tfrac{1}{2}(b + c) \end{aligned}$$

He offers two algebraic solutions for this indeterminate problem. The second one is the most modern one, since he only manipulates equations and not polynomials. But the first solution has an interesting aspect, because we could

[16] Cardano, *Opera* IV, 242: "Exemplum tertium fatis accommodatum. Invenias tres quantitates quarum prima cum secunda sit sequialtera primae cum tertia et prima cum tertia sit sequialtera 2 cum tertia".

call it a derivation with two and a half unknowns. Cardano uses *positio* for the third number and *quantitates* for the second, for which we will use x and y. The sum of the first and third thus is

$$1\frac{1}{2}(x+y).$$

Subtracting the third gives the value of the first as

$$\frac{1}{2}x+1\frac{1}{2}y.$$

Multiplying the sum of the first and third with $1\frac{1}{2}$ gives the sum of first and second as

$$2\frac{1}{4}x+2\frac{1}{4}y.$$

Subtracting the second gives a second expression for the first as

$$2\frac{1}{4}x+1\frac{1}{4}y.$$

As these two are equal

$$1\frac{3}{4}x=\frac{1}{4}y \text{ or } y \text{ is equal to } 7x$$

Only then, Cardano removes the indeterminism by posing that $x = 1$ leading to the solution (11, 7, 1). The interesting aspect of this fragment is that Cardano tacitly uses a third unknown which gets eliminated. As a demonstration, the reasoning can be reformulated in modern notation, with z as third unknown as follows:

$$z+y=1\frac{1}{2}(z+x) \tag{2.3}$$

$$z+x=1\frac{1}{2}(x+y) \tag{2.4}$$

If we subtract x from (2.4) it follows that

$$z=\frac{1}{2}x+1\frac{1}{2}y$$

Substituting (2.4) in (2.3) gives

$$z+y=2\frac{1}{4}x+2\frac{1}{4}y$$

Subtracting y from this equation gives

$$z = 2\frac{1}{4}x + 1\frac{1}{4}y$$

Therefore

$$\frac{1}{2}x + 1\frac{1}{2}y = 2\frac{1}{4}x + 1\frac{1}{4}y$$

or

$$y = 7x$$

There is only a small difference between Cardano's solution and our reformulation. If only he had a symbol or alternative name for the first unknown quantity, it would have constituted an operational unknown. He seems to be aware from the implicit use of three unknowns as he concludes: "And this is a nice method because we are working with three quantities" ("Et est pulchrior modus quia operamur per tres quantitates") (*Opera*, IV, 242). It is not clear why this problem was not included in the 1545 edition. It could have been added by Cardano as a revision to the Basel edition of 1570.

A third aspect from the *Ars magna*, which reveals some evolution in Cardano's use of multiple unknowns is one of the later chapters, describing several rules, previously discussed in the *Practica aritmeticae*. Chapter 31 deals with the *Regula magna*, probably one of the most obscure chapters in the book. The rule is not described, only some examples are given. Nor does it contain any explanation why it is called *The Great Rule*. Most of these problems concern proportions which are represented by letters. Remarkably, Cardano performs operations on these letters and constructs equations using the letters such as "igitur 49 b, aequalia sunt quadrato quadrati a" (see Table 2.2). Only in the final step, as a demonstration that this solves the problem, does he switch back to regular unknown called *res*. Let us look in detail at problem 10 (Witmer 190, *Opera* IV, 276). A modern formulation of the problem is:

$$a + b = 8$$
$$\frac{a^3}{7b} = \frac{7b}{ab}$$

The text is probably the best illustration that the straightforward interpretation of the letters as unknowns is an oversimplification. If the letters would be unknowns then substituting $b = 8 - a$ in $a^4b = 49b^2$ would immediately lead to the equation. Instead, Cardano takes a detour by introducing c, d and e and then applying the magical step 5. No explanation is given, though the inference

$$\frac{a}{7} = \frac{d}{c} \text{ is correct, because } \frac{d}{c} = \frac{7b}{a^3} = \frac{a}{7}, \text{ or } \frac{7b}{a^3} = \frac{ab}{7b}$$

	Symbolic	Meta description	Original text
1	$c = a^3$	choice of unknown	Sit a minor, eius cubus c, b autem maior,
2	$ab = e$	choice of unknown	et productum b in a sit e,
3	$7b = d$	choice of unknown	et septuplum b sit d,
4	$\frac{a}{7} = \frac{e}{d}$	divide (2) by (3)	quia igitur ex b in a, sit e et ex b in 7 sit d, erit a ad 7,
5	$\frac{a}{7} = \frac{d}{c}$		ut e ad d quare a ad 7 ut d ad c
6	$ac = 7d$	multiply (5) by $7c$	Igitur ex a in c, sit septuplum d
7	$a^4 = 49b$	subtitute (1) & (3) in (6)	sed est septuplum b, igitur 49 b aequalia sunt quadrato quadrati a
8	$b = \frac{1}{49}a^4$	divide (7) by 49	igitur b est aequale 1/49 quad. quadrati a
9	$a + b = 8$	premise	quia igitur a cum b est 8
10	$a + \frac{1}{49}a^4 = 8$	substitute (8) in (9)	et b est 1/49 quad. quadrati a, igitur a cum 1/49 quad. quadrati sui, aequatur 8.
11	$x + \frac{1}{49}x^4 = 8$	substitute a by x in (10)	quare res et 1/49 [quad. quadratum aequatur 8]
12	$x^4 + 49x = 392$	multiply (11) by 49	[Igitur] quad. quadratum p. 49 rebus, aequatur 392

Table 2.2: Cardano's *Regula magna* for solving linear problems

which is the reciprocal of what was given. Apparently, the fact that e is to d as d is to c, is evident to Cardano, shows how his reasoning here is inspired by proportion theory, rather than being symbolic algebra.

2.6 The improved symbolism by Stifel

From the last part of Stifel's *Coss* (1553, f. 480^r) we know that he has read the *Ars magna*. He cites Cardano on the discovery of Scipio del Ferro (f. 482^r) and adds a chapter on the cubic equation. The influence between Cardano and Stifel is therefore bidirectional. At several instances he discusses the second unknown from a methodological standpoint, as Cardano did in the *Ars magna*. Although Rudolff does use the second unknown in the original 1525 edition for several problems, in other examples Stifel recommends the *regula quantitatis* as a superior method to the ones given by Rudolff ("Christoff setzet vier operation oder practicirung auff diss exemplum. Ich will eine setzen ist besser und richtiger zu lernen und zu behalten denn seyne vier practicirung", 223v). He notes that there is nothing magical about the second unknown. For him, it is basically not different from the traditional *coss*: "Den im grund ist regula Quantitatis nichts anders denn Regula von 1𝓍." (Stifel 1553, ff. 223^v – –224^r). While we can only wonder why it has not been done before, for Stifel it seems natural to use multiple unknowns for the typical shares or values expressed in linear problems: "Man kan auch die Regulam (welche sye nennen) Quantitatis nicht besser verstehn den durch sollische exempla [i.e. linear problems] Weyl sye doch nichts anders ist denn da man 1 𝓍 setzt under einem andern zeychen" (Stifel 1553, f. 277^v). He considers arithmetical operations on shares not fundamentally different from algebraic operations on unknowns: "Der Cossischen zeychen halb darffest du dich auch nicht hart bekumern. Denn wie 3 fl. un 4 fl. machen 7 fl., also auch 3𝓍 und 4𝓍 machen 7𝓍" (1553, f. 489^r).

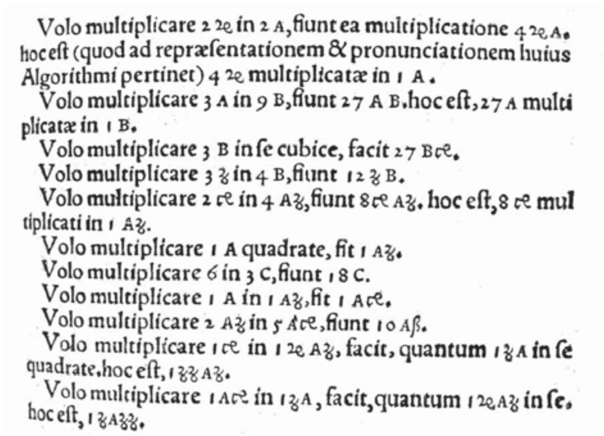
After treating over 400 problems from Rudolff, Stifel adds a chapter with some examples of his own. Half of the 24 problems added are solved by two unknowns. Interestingly, he silently switches to another notation system for quadratic problems involving multiple unknowns, thus avoiding the ambiguities of his original system. The improved symbolism is well illustrated with the following example:[17]

> Find two numbers, so that the sum of both multiplied by the sum of their squares equals 539200. However, when the difference of the same two numbers is multiplied by the difference of their squares this results in 78400. What are these numbers?

This is a paraphrase of Stifel's solution: Using 1𝓍 and $1A$ for the two numbers, their sum is 1𝓍 + $1A$. Their difference is 1𝓍 – $1A$. Their squares 1𝔷 and $1AA$. The sum of the squares 1𝔷 + $1AA$. The difference between the squares 1𝔷 – $1AA$. So multiplying 1𝓍 + $1A$ with 1𝔷 + $1AA$ gives 𝔠 + 1𝔷A + 1𝓍AA + $1AAA$ which equals 539200. Then I multiply also 1𝓍 – $1A$. with 1𝔷 – $1AA$. This gives 𝔠 – 1𝔷A – 𝓍AA + $1AAA$ and that product equals 539200.

[17] Stifel 1553, ff. $469^r - 470^v$, translation mine.

So Stifel now uses AA for the square and AAA for the third power of A. He thus eliminates the ambiguities discussed before. Now that Aℨ becomes AA, the product of the square of A with 1ꝛ can be expressed as AAꝛ and the product of the square of 1ꝛ with A as Aℨ or ℨA – thus also removing the ambiguity of multiplying cossic terms together. As such, algebraic symbolism is functionally complete with respect to to the representation of multiple unknowns and powers of unknowns. What is still missing, as keenly observed by Serfati (2010), is that this does not allow to represent the square of a polynomial. In order to represent the square of 1ℨ + 1ꝛ + 2, for example, Stifel has to perform the calculation. Also, the lack of symbols for the coefficients does not yet allow that every expression of seventeenth-century Cartesian algebra can be written unambiguously in Stifel's symbolism. This was later introduced by Viète. However, the important improvement by Stifel in his *Coss*, was an important step necessary for the development of algebraic symbolism, and has been overlooked by many historians.[18] Having shown that Stifel resolved the ambiguities in the interpretation of multiplied cossic terms, we will further replace the cossic signs for *coss*, *census* and cube by x, x^2 and x^3.



Volo multiplicare 2 ꝛ in 2 A, fiunt ea multiplicatione 4 ꝛ A. hoc eſt (quod ad repræſentationem & pronunciationem huius Algorithmi pertinet) 4 ꝛ multiplicatæ in 1 A.
Volo multiplicare 3 A in 9 B, fiunt 27 A B. hoc eſt, 27 A multiplicatæ in 1 B.
Volo multiplicare 3 B in ſe cubice, facit 27 Bce.
Volo multiplicare 3 ℨ in 4 B, fiunt 12 ℨ B.
Volo multiplicare 2 ce in 4 Aℨ, fiunt 8 ce Aℨ. hoc eſt, 8 ce multiplicati in 1 Aℨ.
Volo multiplicare 1 A quadrate, fit 1 Aℨ.
Volo multiplicare 6 in 3 C, fiunt 18 C.
Volo multiplicare 1 A in 1 Aℨ, fit 1 Ace.
Volo multiplicare 2 Aℨ in 5 Ace, fiunt 10 Aß.
Volo multiplicare 1 ce in 1 ꝛ Aℨ, facit, quantum 1 ℨ A in ſe quadrate. hoc eſt, 1 ℨℨ Aℨ.
Volo multiplicare 1 Ace in 1 ℨ A, facit, quantum 1 ꝛ Aℨ in ſe. hoc eſt, 1 ℨ Aℨℨ.

Fig. 2.3: The improved symbolism by Stifel (1553, f. 469^r)

Next, Stifel eliminates terms from the equation by systematically adding, subtracting, multiplying and dividing the equations, not seen before in his *Arithmetica Integra* of 1544 (Stifel 1553, 469^v):

[18] The symbolism introduced by Stifel in the *Arithmetica integra* is discussed by Bosmans (1905-6), Russo (1959), Tropfke (1980, 285, 377), Gericke (1992, 249-50), Cifoletti (1993) chapter 3, appendix 1 and 2. With the exception of Cajori (1928-9, I, 144-146) who mentions Stifel's innovation as "another notation", none of these authors discuss the significance of the

Multiply the two equations in a cross as you can see below:

$$x^3 + 1x^2A + 1xAA + 1AAA = 539200$$
$$x^3 - 1x^2A - 1xAA + 1AAA = 78400$$

But dividing these numbers by their GCD ("yhre kleynste zalen") gives 337 and 49 and so we arrive at the two sums:

$$49x^3 + 49x^2A + 49xAA + 49AAA$$
$$337x^3 - 337x^2A - 337xAA + 337AAA$$

and these two sums are equal to each other. If we now add $337x^2A + 337xAA$ to each side so, this result in

$$337x^3 + 337AAA = 49x^3 + 386x^2A + 386xAA + 49AAA$$

Now subtract $49x^3 + 49AAA$ from each side, this will give

$$386x^2A + 386xAA = 288x^3 + 288AAA$$

Divide each side by $2x + 2A$, this results in

$$193xA = 144x^2 - 144xA + 144AA \quad (2.5)$$

Next (as you can extract the square root from each side) subtract from each side $144xA$

$$49xA = 144x^2 - 288xA + 144AA$$

Extract from each side the square root, which becomes $\sqrt{49xA} = 12x - 12A$. This we keep for a moment.

Here, operations on equations are remarkably extended to root extraction. Although not fully correct, this can be considered a 'natural' step from previous extensions. Because the alternative solutions are imaginary they are not recognized as such. Only in the seventeenth century we will see the full appreciation of double solutions to quadratic equations. Now Stifel returns to the equation (2.5) ("Ich widerhole yetzt die obgesetzte vergleychung").

Add to each side [of this equation] as much as is needed to extract the root of each side. This is 3 times $144xA$, namely $432xA$. So becomes

$$144x^2 + 288xA + 144A^2 = 625xA$$

Extract again from each side the square root, so will be

$$\sqrt{625xA} = 12x + 12A$$

And before I have found that $\sqrt{49xA} = 12x - 12A$. From these two equations I will make one through addition. Hence

---

improvements of 1553. Eneström (1906-7, 55) spends one page on the improved symbolism discussing Cantor's *Vorlesungen* (1892, 441, 445).

$$24x = \sqrt{1024xA}$$

Next I will square each side, which results in $576x^2 = 1024xA$ and then I divide each side with $576x$. Thus

$$1x = 1\frac{7}{9}A \text{ or } 1A = \frac{9}{16}x$$

Having formulated both unknowns in terms of the other, one of them can be eliminated, or in Stifel's wording *resolved*. He reformulates the original problem in x and 9/16 x, which leads to a cubic expression with solution 64.

We have previously shown that Cardano's operations on equations are implicit in the illustrations but are not rhetorically phrased as such. In this text by Stifel we have a very explicit reference to the construction from one equation by the addition of two others: "From these two equations I make one equation by addition" ("Aufs desen zweyen vergleychungen mach ich ein einige vergleychung mit addiren"). This is certainly an important step forward from the *Arithmetica Integra*, and from then on, operations on equations will be more common during the sixteenth century.

We have here an unique opportunity to compare two works, separated by a decade of development in Stifel's conceptions of algebra. It gives us a privileged insight into subtle changes of the basic concepts of algebra, in particular that of a symbolic equation. As an illustration, let us look at one problem with three numbers in geometric progression. The same problem is presented in Latin in the *Arithmetica Integra* and in German in the Stifel edition of Rudolff's *Coss*, though with different values. The problem is solved using two unknowns in essentially the same way, but there are some delicate differences which are very important from a conceptual point of view. As Stifel presents the problem in a section with "additional problems by his own", we can assume that he constructed the problem himself. In any case, it does not appear in previous writings. In modern formulation the problem has the following structure:

$$\begin{aligned}
&a : b = b : c \\
&(a + c)(a + c - b) = d \\
&(a + c - b)(a + b + c) = e
\end{aligned}$$

with respectively (4335, 6069) and (90720, 117936) for d and e. The start of the solution is identical in the Latin and German text, except that the choice of the first and second unknowns are reversed (see Table 2.3).

In both cases Stifel arrives at two equations in two unknowns. These compares very well with those from Fond. prin. V.152 and the example of Cardano's *Ars Magna*, except that we now have a quadratic expression. If we swap back the two unknowns in the German text, the equations compare as follows:

Stifel 1544, f. 313r	Problem 24, Stifel 1553, f. 474r
Quaeritur tres numeri continue proportionales, ita ut multiplicatio duorum extremorum, per differentiam, quam habent extremi simul, ultra numerum medium, faciant 4335. Et multiplicatio eiusdem differentiae, in summam, omnium trium faciat 6069.	Es sind drey zalen continue proportionales so ich das aggregat der ersten, und dritten, multiplicir mit der differentz dess selbigen aggregatis uber die mittel zal, so kommen 90720. Und so ich die selbige differentz multiplicir in die summa aller dreyer zalen, so kommen 117936. Welche zalen sinds?
1A + 1x est summa extremorum 1A – 1x est summa medij $2A$ est summa omnium trium $2x$ est differentia quam habent extremi ultra medium.	Die drey zalen seyen in einer summa 2x. Die zurlege ich also in zwo summ $1x + 1A$, $1x - 1A$ Nu last ich $1x - 1A$ die mittel zal seyn so muss $1x + 1A$ die summa seyn der ersten und dritten zalen. Und also sind $2A$ die differentz dess selbigen aggregats uber die mittel zal.
Itaque $2x$ multiplicatae in summam extremorum, id est, in 1A + 1x faciunt $2xA + 2x2$ aequata 4335.	Drumb multiplicir ich $2A$ in $1x + 1A$ facit $2xA + 2AA$ gleych 90720.
Deinde $2x$ multiplicatae in $2A$ seu in summam omnium, faciunt $4xA$ aequata 6096.	So ich aber $2A$ multiplicir in die summ aller dreyer zalen, nemlich in $2x$, so kommen $4xA$ die sind gleych 117936.

Table 2.3: Two ways how Stifel solves structurally the same problem.

$$2xy + 2x^2 = 4335 \qquad 2xy + 2x^2 = 90720$$
$$4xy = 6096 \qquad 4xy = 117936$$

The next step is to eliminate one unknown from the two equations. We have seen that Cardano was the first to do this by multiplying one equation to equal the coefficients of one term in both equations and then to subtract the equations, albeit implicitly. In this respect, the later text deviates from the former (see Table 2.4).

The method in the Latin text articulates the value of xy from the two expressions and *compares* the resulting values. The text only states that their values are equal. Although Stifel writes "Confer iam duas aequationes illas", this should be understood as "now match those two equal terms", *aequationes* being the acts of comparing. So from the first expression we can infer that the value of xy is $(4335 - 2x2)/2$. From the second we can know that the value is 6069/4. Thus, $(4335 - 2x2)/2$ must be equal to 6069/4, from which we can deduce the value of x. The reasoning here is typical for the abacus and early cossist tradition were the solution is based on the manipulation and equation of polynomials expressions. In the German text, a decade later, Stifel distinctly moves to the manipulation of equations. He literally says: "Now double the equation above" and "from this [equation] I will now subtract the numbers

Stifel 1544, f. 313r	Problem 24, Stifel 1553, f. 474v
Confer iam duas aequationes illas. Nam ex priore sequitur quod $1xA$ faciat $(4335 - 2x^2)/2$.	So duplir ich nu die obern vergleychung, fa. $4xA + 4AA$ gleych 181440.
Ex posteriore autem sequitur quod $1xA$ faciat 6069/4. Sequitur ergo quod $(4335 - 2x^2)/2$ et 6069/4 inter se aequentur. Quia quae uni et eidem sunt aequalia, etiam sibi invicem sunt aequalia. Ergo (per reductionem) $17340 - 8x^2$ aequantur 12138 facit $1x^2 \cdot 650\frac{1}{4}$.	Da von subtrahir ich yetzt die zalen diser yetzt gefundnen vergelychung. Nemlich $4xA$ gleych 117936 so bleyben $4AA$ gleych 63504.
Et $1x$ facit $25\frac{1}{2}$.	Also extrahir ich auff yeder seyten die quadrat wurzel, so werden $2A$ gleych 252 und ist die differentz dess aggregats uber die mittel zal. So in nu $1A$ gleych 126.

Table 2.4: Two ways how Stifel solves structurally the same problem.

of the newly found equation", thus eliminating the second unknown. The last step also shows a clear evolution. In the Latin text he reduces the expression to the square of the unknown $1x^2$ and then extracts the root. In the later text he "extracts the square root of each side [of the equation]". The rest of the problem is to reformulate the original problem using the value of the second unknown. This is done in similar ways.

The example shows how the road to the concept of a symbolic equation is completed in a crucial decade of algebraic practice of the mid-sixteenth century. We have witnessed this evolution within a single author. The French algebraists from the second half of the sixteenth century will extend this evolution to a system of simultaneous linear equations.

2.7 Towards an aggregate of equations by Peletier

Stifel's edition of the *Coss* was published in Köningsberg in 1553, his foreword is dated 1552. Peletier's postscript ends the *Algèbre* with the date July 28, 1554. The printer's permit allows him to print and sell the book for three years from June 15, 1554. So, while Peletier might have seen Stifel's edition of the *Coss*, it does not show in his book. He certainly has studied the *Arithmetica Integra* well.

Jacques Peletier spends one quarter of the first book on the second unknown which he calls *les racines secondes* (pp. 95-117), a direct translation of Stifel's *de secundis radicibus* (Stifel 1544, f. 251v). He introduces Stifel's notation by way of the problem of finding two numbers, such that, in modern formulation (Peletier 1554, 96):

$$x^2 + y^2 = 340$$
$$xy = \frac{6}{7}x^2$$

If we would use the same name for the unknown for both numbers, this would lead to confusion, he argues. He therefore adopts Stifel's notation of $1A$, $1B$ for the second and third unknown in addition to his own sign for the first unknown. He then discusses the operations with multiple unknowns: addition, subtraction, multiplication and division, as was done with polynomials in his introductory chapters. He retains Stifel's ambiguity from the *Arithmetica Integra* that xy cannot be differentiated from yx.
Peletier has selected this example, instead of the one used by Stifel, because that problem can easily be solved in one unknown ("Car il est facile par une seule posicion sans l'eide des secondes racines", Peletier 1554, 102).

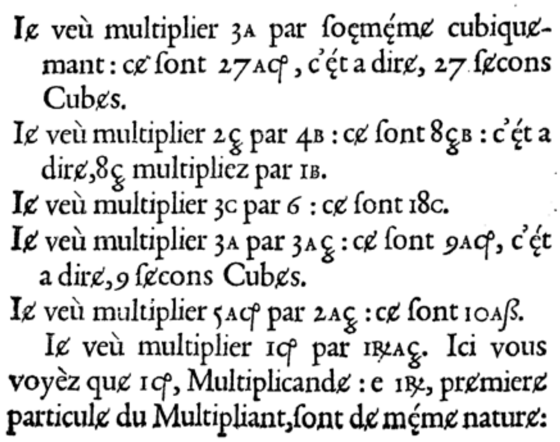

Ie veù multiplier 3A par ſoęmęme cubiquemant : ce ſont 27Acꝭ, c'ęt a dire, 27 ſecons Cubes.
Ie veù multiplier 2ç par 4B : ce ſont 8çB : c'ęt a dire, 8ç multipliez par 1B.
Ie veù multiplier 3C par 6 : ce ſont 18C.
Ie veù multiplier 3A par 3Aç : ce ſont 9Acꝭ, c'ęt a dire, 9 ſecons Cubes.
Ie veù multiplier 5Acꝭ par 2Aç : ce ſont 10Aß.
Ie veù multiplier 1cꝭ par 1B℞Aç. Ici vous voyèz que 1cꝭ, Multiplicande : e 1B℞, premiere particule du Multipliant, ſont de męme nature:

Fig. 2.4: The rules for multiplying terms with multiple unknowns from Peletier (1554, 98). Compare these with Stifel (1545, f. 252^r)

Using x for the larger number and y for the smaller one he squares the second equation to

$$x^2y^2 = \frac{36}{49}x^4 \text{ which leads to } 49y^2 = 36x^2$$

Because $y^2 = 340 - x^2$ this can be rewritten as $y^2 = 340 - \frac{49}{36}y^2$. Then the second unknown can be expressed as

$$2\frac{13}{36}y^2 = 340 \text{ or } y^2 = 144,$$

leading to the solution 12 and 14.

Peletier gives four other problems solved with multiple unknowns. The first two are taken from Cardano's *De Quaestionibus Arithmeticis* in the *Practica arithmeticae*, problem 97 and 98 (Cardano, Opera III, 168-9), the third is the problem from Cardano's *Ars magna* discussed above (2.2). The fourth is one from Stifel (1544, f. 310^v), reproducing the geometric proof. This shows that Peletier was well acquainted with the most important algebraic treatises of his time. In fact, Peletier's example III (1554, 105-7) and its solution, is a literal translation from Cardano's, only using the symbolism from Stifel. The problem is structurally similar to problem 41 from Pacioli discussed earlier and follows the method by Pacioli. Compare the following text fragments:

Cardano, 1539, ff. HH.vir - HH.viv	Peletier, 1554, p. 106
Igit*ur* per p*raecedente*m iunge summa*m* eoru*m* sit 3 quan. m. 31/30 co. divide p*er* 1 m. numero hominu*m* q*uod* est 2 exit 1 quan. m. 31/60 co. et haec est summa quae debet aequari valori equi sed aequus valet 1. quan. igit*ur* 1 quan. m. 31/60 co. aequantur 1 quan. quare detrahe 1 quan. ex 1 quan. remanebit quan. equivale*ns* 31/60 co. igitur 1 quan. aequivalet duplo q*uod* est 31/30 co. igitur dabis ex hoc fracto valore*m* denominatoris q*ui* e*st* 30 [sic] ad co. et numeratore*m* ad quan. igitur valor co. e*st* 30 et valor qua*n*titatis est 31 et in bursa fuere 30.	Par la precedente, assemblez les troes sommes: ce sont 3A m. 31/30 R. Divisez par un nombre moindre de 1 que les hommes, savoer est par 2: ce sont 1 A m. 31/60 R. E c'est la valuer du cheval. Donq, 1A est egale a 1 A m. 31/60 R. E par souttraction, A est egale a 31/60 R. Donc 1A, vaut la double, qui est 31/30 R. Meintenant, prenez pour 1A, le numerateur, que est 31, e pour 1R prenez le denominateur 30. Partant, le cheval valoet 31 e l'argant commun etoest 30.

Table 2.5: The dependence of Peletier on Cardano's *Practica Arithmeticae*.

Peletier thus literally translated Cardano's text only changing 1 *quan.* in $1A$ and reformulating the common sum as the value of a horse. We included this fragment to show how strongly Peletier bases his algebra on Cardano while Cifoletti attributes to him an important role in the development towards a symbolic algebra. Nonetheless, Peletier introduces some interesting new aspects in the next linear problem taken from *Ars Magna*. He first gives a literal translation of Cardano's solution calling the problem text *proposition* and the solution *disposition*. Interestingly he leaves out the substitution steps from Cardano, lines (7) and (14). Cardano considered these important for a demonstration, but apparently Peletier does not. Then he introduces a solution of his own ("trop plus facile que l'autre"). Starting from the same formulation (2.2), Peletier adapts Cardano's solution method by means of Stifel's symbolism for multiple unknowns.

	Symbolic	**Meta description**	**Original text**
1	$a = x$	choice of first unknown	Le premier à 1R
2	$b = y$	choice of second unknown	Le second 1A
3	$c = z$	choice of third unknown	Le tiers 1B.
4	$a + \frac{1}{2}(b + c) = 32$	premise	E par ce que le premier avec $\frac{1}{2}$ des deus autres, an à 32:
5	$x + \frac{1}{2}(y + z) = 32$	substitute (1), (2) and (3) in (4)	1R p. (1A p. 1B)/2 seront egales a 32.
6	$2x + y + z = 64$	multiply (5) by 2	E par reduccion, e due transposicion: 2R p. 1A p. 1B sont egales a 64, qui sera la premiere equacion.
7	$b + \frac{1}{3}(a + c) = 28$	premise	Secondemant, par ce que le second, avec 1/3 partie des deus autres an à 28:
8	$y + \frac{1}{3}(x + z) = 28$	substitute (1), (2) and (3) in (6)	ce sont 1A p. (1R p. 1B)/3 egales a 28:
9	$x + z + 3y = 84$	multiply (8) by 3	E par reduccion, 1A p. 1B p. 3A seront egales a 84, qui sera la seconde equacion.
10	$c + \frac{1}{4}(a + b) = 31$	premise	Pour le tiers (lequel avec $\frac{1}{4}$ partie des deus autres an à 31),
11	$z + \frac{1}{4}(x + y) = 31$	substitute (1), (2) and (3) in (10)	nous aurons 1B p. (1R p. 1A)/4, egales a 31.
12	$x + y + 4z = 124$	multiply (11) by 4	e par samblable reduccion, 1R p. 1A p 4B seront egales a 124. Voela, noz troes equacions principales.

Table 2.6: Peletier solving a problem by multiple unknowns.

Having arrived at three equations in three unknowns there seems to be little innovation up to this point. All operations and the use of three unknowns have been done before by Stifel. However, we can discern two subtle differences. Firstly, the last line (12) suggests that Peletier considers the three equations as an aggregate. In the rest of the problem solving process he explicitly acts on this aggregate of equations ("disposons donq nos troes equacions an cete sorte"). Secondly, he identifies the equations by a number. In fact, he is the first one in history to do so, a practice which is still in use today.[19] The identification of equations, as structures which you can manipulate, facilitates the rhetorical structure of the *disposition*. This becomes evident in the final part (see Table 2.7).

	Symbolic	**Meta description**	**Original text**
13	$2x + 4y + 5z = 208$	add (9) and (12)	Ajoutons la seconde e la tierce, ce seront, pour quatrieme equacion 2R p. 4A p. 5B egales a 208
14	$3y + 4z = 144$	subtract (6) from (13)	Donq an la conferant a la premier equacion, par ce que 2R sont tant d'une part que d'autre, la differance de 64 a 208 (qui est 144) sera egale avec la differance de 1A p. 1B a 4A p. 5B: Donq, an otant 1A p. 1B de 4A p. 5B, nous aurons pour la cinquieme equacion 3A p. 4B egales a 144
15	$3x + 4y + 2z = 148$	add (6) and (9)	ajoutons la premiere e la seconde: nous aurons pour la sizieme equacion 3R p. 4A p. 2B egales a 148.
16	$3x + 2y + 5z = 188$	add (6) and (12)	ajoutons la premiere e la tierce: nous aurons pour la sesttieme equacion 3R p. 2A p. 5B egale a 188.
17	$6x + 6y + 7z = 336$	add (15) and (16)	ajoutons ces deus dernieres: nous aurons, pour la huitieme equacion 6R p. 6A p. 7B egales a 336.
18	$6x + 6y + 24z = 744$	multiply (12) by 6	Finablemant, multiplions la tierce par 6 (pour sere les racines egales, de ces deus dernieres equacions) e nous aurons, pour la neuvieme equacion 6R p. 6A p. 24B egales a 744.

Table 2.7: Peletier eliminating unknowns by adding and subtracting equations.

Peletier succeeded in manipulating the equations in such a way that he arrives at two equations in which two of the unknowns have the same coefficients, or in his terms, "equal roots". Subtracting the two gives $17z = 408$ arriving at the value 24 for z. The other values can then easily be determined as 12 and 16. Comparing his method with Cardano's, it is not shorter or more concise. Cardano takes 16 steps to arrive at two equations in which one unknown can be eliminated, Peletier takes 18 steps to the elimination of two unknowns. But

[19] The classic work by Cajori (1928-9) on the history of mathematical notations, does not include the topic of equation numbering or referencing. I have seen no use of equation

Peletier does not use the argument of length, instead he considers his method easier and clearer, thus emphasizing the argumentative structure. Indeed, as can be seen from the table, the actual text fits our meta-description very well. Peletier systematically uses operations on equations and applies addition and subtraction of equations to eliminate unknowns. Moreover, he explicitly formulates the operations as such: "add the second [equation] to the third, this leads us to a fourth equation". Although we have seen such operations performed implicitly in Cardano's illustration, the use of the terminology in the argumentation is an important contribution. The use of multiple unknowns, the symbolism and the argumentation, referring to operations on structures, called equations, makes this an important entrance into symbolic algebra.

2.8 Valentin Mennher (1556)

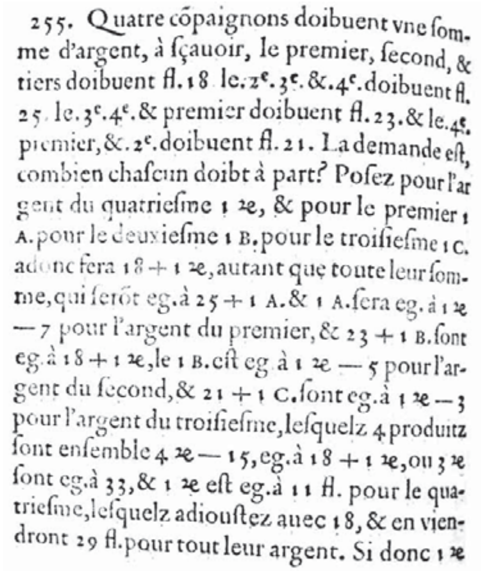
Valentin Mennher, a reckoning master from Antwerp, introduces the rule in between problems 254 and 255 as *regle de la quantité, ou seconde radice* in his *Arithmétique seconde* (Mennher, 1556, f. Qi^v; 1565, f. FFi^r) as a "rule which exceeds all other rules and without which many examples would otherwise be unsolvable". He refers to Stifel for the origin of the rule and adopts Stifel's notation.[20] From problem 267, it becomes clear that he has used Stifel's edition of Rudolff (1553) as he also uses the improved notation AA for the square of the second unknown (1556, ff. $Qvi^r - Qvi^v$; 1565, ff. $Ffviii^r - Ffviii^v$). We will give one example from Mennher, though the method does not differ from Stifel's solution to problem 193 of Rudolff's *Coss*. The problem is about four persons having a debt, with the four sums of three given. The problem is known from early Indian sources. Stifel uses four unknowns while Rudolff originally reuses the second unknown. Mennher adopts Stifels method with different values and slightly changing the unknowns. Mennher uses the values:

$$\begin{aligned} a+b+c &= 18 \\ b+c+d &= 25 \\ a+c+d &= 23 \\ a+b+d &= 21 \end{aligned}$$

With the unknowns x, A, B and C for d, a, b and c respectively, he expresses the sum of all four as $18 + x$, $25 + A$, $23 + B$ and $21 + C$.

numbers prior to Peletier's.

[20] Mennher, clearly learned the use of letters from Stifel, as he writes: "tout ainsi comme M. Stiffelius l'enseigne, en posant apres le x pour la seconde position A, et pour la troisiesme B, et pout la quatriesme C." (Mennher, 1556, Qi^v; 1565 $Ffi^r - Ffi^v$).



255. Quatre cõpaignons doibuent vne ſom-
me d'argent, à ſçauoir, le premier, ſecond, &
tiers doibuent fl. 18 le. 2^e. 3^e. &. 4^e. doibuent fl.
25. le. 3^e. 4^e. & premier doibuent fl. 23. & le. 4^e.
premier, &. 2^e. doibuent fl. 21. La demande eſt,
combien chaſcun doibt à part? Poſez pour l'ar
gent du quatrieſme 1 ℞, & pour le premier 1
A. pour le deuxieſme 1 B. pour le troiſieſme 1 C.
adonc ſera 18 + 1 ℞, autant que toute leur ſom-
me, qui ſerõt eg. à 25 + 1 A. & 1 A. ſera eg. à 1 ℞
— 7 pour l'argent du premier, & 23 + 1 B. ſont
eg. à 18 + 1 ℞, le 1 B. eſt eg. à 1 ℞ — 5 pour l'ar-
gent du ſecond, & 21 + 1 C. ſont eg. à 1 ℞ — 3
pour l'argent du troiſieſme, leſquelz 4 produitz
ſont enſemble 4 ℞ — 15, eg. à 18 + 1 ℞, ou 3 ℞
ſont eg. à 33, & 1 ℞ eſt eg. à 11 fl. pour le qua-
trieſme, leſquelz adiouſtez auec 18, & en vien-
dront 29 fl. pour tout leur argent. Si donc 1 ℞

Fig. 2.5: The use of the second unknown by Mennher (1556, f. Ffi^r).

As these four expressions have the same value, the debts of the first three can be restated in terms of x, namely $x - 7$, $x - 5$, and $x - 3$ respectively.

Adding the three together with x leads to the sum of all four $4x - 15$, which is equal to $18 + x$. From this it follows that x is 11, and the other debts are 4, 6 and 8. Most of the last twenty problems in the book are solved using several unknowns.

2.9 Kaspar Peucer (1556)

The humanist Caspar Peucer wrote, among other works on medicine and philosophy, a Latin algebra with the name *Logistice Regulae Arithmeticae*. The book contributed little to the works published by Stifel and had little influence. Except for a recent paper by Mei*β*ner and Deschauer (2005), Peucer seems to be forgotten. He discusses the *regula quantitatis* by the term *radicibus secundis* and provides four examples (Peucer 1556, ff. Tvir-Viir). He refers to Rudolff, Stifel and Cardano for the origin of the method. His first example is the ass and mule problems from the Greek epigrams, creating the indeterminate equation $1x{+}1 = 1A{-}1$. The other problems are linear ones involving multiple unknowns. The symbolism is taken from Stifel (1544).

2.10 Towards a system of simultaneous equations

2.10.1 Buteo (1559)

Jean Borel, better known under his Latinized name Buteo, is an underestimated as an author of mathematical works during the sixteenth century. He started publishing only after he became sixty. His *Logistica* of 1559 is a natural extension of the ideas of Peletier. Though Peletier was the first to consider an aggregate of equations, Buteo improved on Peletier and raised the method to what we now call solving a system of simultaneous linear equations. The naming of his book by the Greek term of *logistics* is an implicit denial of the Arab contributions to Renaissance algebra. This position is shared by several humanist writers of the sixteenth century.

Buteo introduces the second unknown in the third book on algebra in a section *De regula quantitatis* (Buteo 1559, f. 189^r). For the origin of the rule he cites Pacioli and de la Roche (by the name Stephano). While the name of the rule is indeed derived from de la Roche, Buteo remains quiet about his main source, his rival Peletier.[21]

After an explanation of the method by means of four examples he solves many linear problems by multiple unknowns in the fifth book. He introduces some new symbols but he had too little influence on his peers to be followed in this. Where Peletier and Mennher still used the radix or cossic sign for the first unknown, Buteo assigns the letter A to the first unknown and continues with B, C, .. for the other unknowns. Ommitting the cossic signs all together, Buteo takes a major step into the "representation of compound concepts", a necessary step towards algebraic symbolism according to Serfati (2010). The next step would be the use of exponents as introduced by Descartes in the *Regulæ*. Buteo further uses a comma for addition, the letter M for subtraction and a left square bracket for an equation. Thus the linear equation

$$6x + 3y + 2c = 84$$

is written as

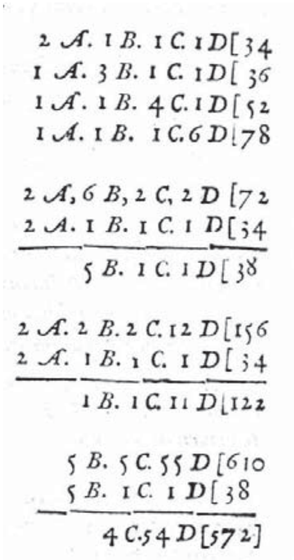
$$6A, 3B, 2C[84$$

[21] Apart from a theoretical dispute on the angles of contact, in which Buteo's *Apologia* of 1562 pursues a refutation of Peletier, there existed a real hostility between them.

Once an equation is resolved in one unknown, he uses two brackets as in

$$5C[60] \text{ for } 5z = 60$$

A fragment of the fourth example is shown in Figure 2.6.


2 A. 1 B. 1 C. 1 D[34
1 A. 3 B. 1 C. 1 D[36
1 A. 1 B. 4 C. 1 D[52
1 A. 1 B. 1 C. 6 D[78

2 A, 6 B, 2 C, 2 D [72
2 A. 1 B. 1 C. 1 D[34
5 B. 1 C. 1 D[38

2 A. 2 B. 2 C. 12 D[156
2 A. 1 B. 1 C. 1 D[34
1 B. 1 C. 11 D[122

5 B. 5 C. 55 D[610
5 B. 1 C. 1 D[38
4 C. 54 D[572]

Fig. 2.6: Systematic elimination of unknowns by Buteo (1559, 194)

Buteo refers to equations, not by numbers as Peletier but at least by their order. As an example let us look at question 30 (Buteo 1559, 357-8). His commentary is very terse (see Table 2.8).

With this and other examples, Buteo systematically manipulates equations to eliminate unknowns. His explanation refers explicitly to the multiplication of equations and the operations of adding or subtracting two equations. The idea of substitution is implicitly present, but is not performed as such, as can be seen from the missing commentaries for steps (13) and (16).

2.10.2 Pedro Nunes criticizing the second unknown

Although from Portugese origin, Nunes wrote his treatise on algebra in Spanish and published it in Antwerp.[22] Because his *Algebra* was published in 1567, it could appear that Nunes did not take advantage of the significant advances

[22] His name is therefore often written in the Spanish form Pedro Nuñez.

	Symbolic	Meta description	Original text
1	$x + \frac{y}{2} + \frac{z}{3} = 14$	premise	Huius solution secundum quantitatis regulam investigabitur, hoc modo. Pone Biremes esse $1A$, Triremes $1B$, Liburnicas $1C$. Erit igitur $1A$, B, $1/3$ C [14. Item $1B$, $1/3$ A, C [13. Et $1C$, $1/6$ A, $1/8$ B [14.
2	$\frac{x}{3} + y + \frac{z}{4} = 13$	premise	
3	$\frac{x}{6} + \frac{y}{8} + z = 14$	premise	
4	$6x + 3y + 2z = 84$	multiply (1) by 6	
5	$4x + 12y + 3z = 156$	multiply (2) by 12	
6	$4x + 3y + 24z = 336$	multiply (3) by 24	
7	$24x + 12y + 8z = 336$	multiply (4) by 4	multiplica aequationem (4) in 4
8	$20x + 5z = 180$	subtract (5) from (7)	auser (5) restat
9	$10x + 15y + 5z = 240$	add (4) and (5)	adde (4) (5)
10	$10x + 60 = 15y$	subtract (9) from (8)	Inter duas equationem postremas que sunt (8) et (9) differentia est (10)
11	$5z = 60$	subtract (10) from (8)	qua sublata ex (10) restat (11)
12	$z = 12$	divide (11) by 5	Partire in 5 provenit (12)
13	$20x + 60 = 180$	substitute in (12) in (8)	
14	$20x = 180 - 60$	resolves (13)	habeas Biremes ex aequatione ubi est 180 auser 60
15	$x = 6$	divide (14) by 20	partire (14) in 20
16	$2 + y + 3 = 13$	substitute (15), (12) in (2)	
17	$y = 8$	resolves (16)	et Trimeres erunt 8
			Quod erat quaesitum.

Table 2.8: Buteo's handling of a system of linear equations.

in symbolic algebra established during the decades before him. However, in the introduction, Nunes explains that he wrote most of the book over thirty years ago.[23] He chose to base much of the problems treated in his book on the *Summa* by Pacioli (1494). He questions some innovations that he learned from Pacioli, such as the use of the second unknown. Nunes discusses the problem of three men comparing their money as treated by Pacioli in distinction 9, treatise 9, paragraph 26 (1494, f. $191^v - 192^r$). However, the values of the problem are not those of Pacioli but are identical to the problem of Cardano, which we discussed above (2.2). Nunes does not reduce the problem

[23] John Martyn discovered a manuscript in 1990, the Cod. cxiii/1-10 at Municipal Library of Évora, Portugal. This Portugese text, written in 1533, contains an algebra which he attributed to Pedro Nunes. The date corresponds well with this thirty years of time difference. Martyn (1996) published an English translation and put much effort in the demonstration of the similarities with the Spanish text of 1567. The attribution of this text to Nunez has recently been refuted by Leitão (2002).

to two linear equations in two unknowns to be resolved by manipulating the equations as did Cardano (1545). Instead he follows the solution method in two unknowns from Pacioli.[24] He then provides a solution of his own, using a single unknown and concludes with the following observation:[25]

> But having treated the same example, that is case 51, we solved this with much ease, and more concise by the single unknown, without the use of the absolute quantity. And all the cases that Father Lucas solved with the [rule of] quantity, we solved by the rules of the unknown, without the aid of this last quantity.

Nunes is not very impressed by the *regula quantitatis* in which others saw "a more beautiful" way for solving problems or even "a perfection of algebra". He believes that most (linear) problems can be solved easier and shorter by a single unknown.

Similar criticism was formulated by other authors. Bosmans discovered a copy of the *Arithmetica Integra* by Stifel (1544) with marginal annotations from Gemma Frisius. The book, kept at the Louvain university library, has unfortunately been destroyed during World War I. Bosmans (1905-6, 168) reports three occasions in which Frisius critizes Stifel for using the second unknown: "Haec quaestio non requirit secundas radices" (f. 252v), "hic quoque secundis radicibus non est opus" (f. 253r), "et haec quastio secundis radicibus non est opus" (f. 253v) and "et haec quaestio secundis radicibus absolve potest" (f. 255r). This demonstrates that the use of the second unknown was still controversial during the mid-sixteenth century.

One could blame Frisius and Nunes for a reactionary view point. Bosmans (1908a, 159) quotes Nunes with some examples in which he rejects negative solutions and zero as a solution to an equation. However, Nunes had a very modern approach to algebra. As pointed out by Bosmans (1908a, 163), he can be credited as being the first who investigates the relationship of the following product with the structure of the equations (Nunes 1567, f. 125^v):

$$(x+1)(x+1), (x+1)(x+2), (x+1)(x+3)\ldots$$
$$(2x+1)(x+1), (2x+1)(x+2), (2x+1)(x+3)\ldots$$

As we now known from further developments, such investigations were important to raise sixteenth century algebra from arithmetical problem solving to the study of more abstract algebraic structures and relations. This leads us to the last author before Viète writing on the *Regula quantitatis*.

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[24] We omit the solution here because a complete transcription of the problem with a symbolic translation is provided by Bosmans (1908b, 21-2).

[25] Nunes 1567, f. 225v: "Pero nos avemos tratado esto mismo exemplo, que es el caso 51, y lo practicamos muy facilmente, y brevemente por la cosa, sin usar de la quantidad absoluta. Y todos los casos que Fray Lucas practica por la quantidad, practicamos nos por las reglas de la cosa, sin ayuda deste termino quantidad".

2.10.3 Gosselin (1577)

Guillaume Gosselin's *De Arte Magna* is our last link connecting the achievements of Cardano, Stifel, and Buteo using the second unknown with Viète's study of the structure of equations in his *Isagoge*. Cifoletti (1993) has rightly pointed out the importance of this French tradition to the further development of symbolic algebra.

Gosselin is rather idiosyncratic in his notation system and seems to ignore most of what was used before him. For the arithmetical operators, addition and subtraction he uses the letters P and M, rather than + and – as was commonly used in Germany and the Low Countries at that time and also adopted by Ramus in France. However, five years later in *de Ratione* (Gosselin, 1583) he did use the + and – sign. The letter 'L' (from *latus*) is used for the unknown; the square becomes 'Q' and the cube 'C'. In some cases he refers to the second unknown by 'q', as did Cardano. For a linear problems with several unknowns he switches to the letters *A*, *B*, C*,* as Buteo, but evidently leading to ambiguities with the sign for x^3. Even more confusing is the use of 'L' for the root of a number, such as

$$\text{L9 for } \sqrt{9} \text{ and LC8 for } \sqrt[3]{8}$$

Accepting isolated negative terms, the letter 'M' is also used as M8L for $-8x$. Gosselin follows Buteo with equations to zero as in '3QM24L aequalia nihilo', for $3x^2 - 24x = 0$ (Gosselin 1577, f. 73^v). The symbolism adopted by Gosselin can be illustrated with an example of the multiplication of two polynomials (ibid. f. 45^v):

	4 L M 6 Q P 7
	3 Q P 4 L M 5
	12 C M 18 QQ P 21 Q
Producta	16 Q M 24 C P 28 L
	M 20 L P 30 Q M 35
Su*m*ma	67 Q P 8 L M 12 C M 18 QQ M 35

The major part of book IV deals with the second unknown, though his terminology is rather puzzling. Chapter II is titled *De quantitate absoluta* (f. 80^r) and chapter III (misnumbered as II) as *De quantitate surda* (f. 84^r). In both these chapters Gosselin solves linear problems with several unknown quantities. So what is the difference? Gosselin gives no clue as he leaves out any definitions of the terms. However, we have previously seen that 'absolute quantity' is used by Nunes and *quantita sorda* by Pacioli and Cardano.[26] From a comparison of the five problems solved by 'absolute quantities' with the four solved by the *quantita surda* it becomes apparent that Gosselin places the distinction between multiple unknowns and the second unknown. Thus the 'absolute quantities' correspond with the symbolic unknowns A, B, C, .. as used by Buteo. Gosselin leaves out the primary unknown of Stifel or Peletier, as was previously done by Buteo. The *quantita surda* corresponds with the *quan.* of Cardano (1545), for which Gosselin uses the symbol q. The *positio* of Cardano becomes the *latus* for Gosselin.

$$\frac{1}{2}y + 2 + x = \frac{9}{2}y - 18$$
$$x + 20 = 4y$$
$$y = \frac{1}{4}x + 5$$

$$\frac{1}{3}x + 3 + \frac{1}{4}x + 5 = 2x - 9$$
$$\frac{17}{12}x = 17$$

$$x = 12, y = 8$$

Sit prior numerus 1 L, ſecundus 1 q, atque ſic $\frac{1}{2}$ q P 2 P 1 L æqualia ſunt reſidui noncuplo nempe $\frac{9}{2}$ q M 18, & addito quod deficit ſubductoque ſuperfluo 1 L P 20 æqualia 4 q, fit 1 q, $\frac{1}{4}$ L P 5, iam prior vt ſupra ſit 1 L, ſecundus erit $\frac{1}{4}$ L P 5, atque adeo $\frac{1}{3}$ L P 3 P $\frac{1}{4}$ L P 5 æqualia 2 L M 9, & addito quod deficit ſubductoque ſuperfluo $\frac{17}{12}$ L æquales 17, fit vnum latus 12, & tantus eſt prior numerus, ſecundus $\frac{1}{4}$ L P 5 hoc eſt 8.

Table 2.9: Gosselin's use of the *quantita surda* (Gosselin, 1577, f. 84v)

Cifoletti (1993, 138-9) concludes on Gosselin that

> it is true that this innovation originates with Borrel [Buteo], but Gosselin uses it with a new skill that permits him to more easily solve the same problems proposed by Borel. It seems reasonable to think that Viète took his symbol as point of departure to arrive at his A, E. Gosselin could also be a source for the notation used by Descartes, who in the *Regulae* proposes to designate the known term with lower-case letters and the unknown with capitals.

[26] Cifoletti (1993, 136) is wrong in claiming that "Cardano does not use the word *surda* in this sense". Furthermore, she translates the *quantita surda* as the surd quantity and speculates on irrational quantities. However, the Italian term *sorda*, as used by Pacioli, means 'mute' in Italian. Thus *quantitate sorda* may simply refer to the voiceless consonant

	Symbolic	Meta description	Original text
1	$x + \frac{y}{2} + \frac{z}{2} + \frac{u}{2} = 17$	premise	1ABCD aequalia 17
2	$\frac{x}{3} + y + \frac{z}{3} + \frac{u}{3} = 12$	premise	1B1/3A1/3C1/3D aequalia 12
3	$\frac{x}{4} + \frac{y}{4} + z + \frac{u}{4} = 13$	premise	1CABD aequalia 13
4	$\frac{x}{6} + \frac{y}{6} + \frac{z}{6} + u = 13$	premise	1D1/6A1/6B1/6C aequalia 13
5	$2x + y + z + u = 34$	multiply (1) by 2	revocentur ad integros numeros, existent 2A1B1C1D aequalia 34
6	$x + 3y + z + u = 36$	multiply (2) by 3	1A3B1C1D aequalia 36
7	$x + y + 4z + u = 52$	multiply (3) by 4	1A1B4C1D aequalia 52
8	$x + y + z + 6u = 78$	multiply (4) by 6	1A1B1C6D aequalia 78
9	$2x + 2y + 5z + 7u = 130$	add (7) and (8)	addamus duas ultimas aequationes, tertiam scilicet et quartam, existent 2A2B5C7D aequalia 130
10	$y + 4z + 6u = 96$	subtract (5) from (9)	tollamus hinc primam, restabunt 1B4C6D aequalia 96
11	$2x + 4y + 2z + 7u = 114$	add (6) and (8)	addamus quartam et secundam, fient 2A4B2C7D aequalia 114
12	$3y + z + 6u = 80$	subtract (5) from (11)	tollamus hinc primam, supererunt 3B1C6D aequalia 80
13	$2x + 4y + 5z + 2u = 88$	add (6) and (7)	addamus secundam et tertiam aequationem, fient 2A4B5C2D aequalia 88
14	$3y + 4z + u = 54$	subtract (5) from (13)	tollamus primam, restabunt 3B4C1D aequalia 54
15	$3y + 12z + 18u = 288$	multiply (10) by 3	iam vero triplicemus 1B4C6D quae fueru*n*t aequalia 96 fient 3B12C18D aequalia 288
16	$11z + 12u = 208$	subtract (12) from (15)	tollamus hinc 3B1C6D aequalia 80, restabunt 11C12D aequalia 20
17	$8z + 17u = 234$	subtract (14) from (15)	subducamus iterum ex eadem triplicata aequatione 3B4C1D eaqualia 54, restabunt 8C17D aequalia 234
18	$88z + 187u = 2574$	multiply (17) by 11	multiplicemus hanc aequationem in 11, fient 88C187D aequalia 2574
19	$88z + 96u = 1664$	multiply (16) by 8	ducamus etiam 11C12D aequalia 208, in 8, existent 88C96D aequalia 1664
20	$91u = 910$	subtract (19) from (18)	tollamus 88C96D aequalia 1664 ex 88C187D *a*equalibus 2574, restabunt 91D aequalia 910 sicque stat aequatio
21	$u = 10$	divide (20) by 91	partiemur 910 in 91, quotus erit 10 valor D, est ergo 10 ultimus numerus ex quaesitis
22	$11z + 120 = 208$	substitute (21) in (16)	et quoniam 11C12D erant aequalia 208,

Table 2.10: Gosselin's solution to a problem from Buteo.

q representing 'quantity'. In English a voiceless consonant is also called a surd.

We believe that the influence on Viète and Descartes attributed to Gosselin by Cifoletti is too much of an honour for Gosselin. The many ambiguities in Gosselin's system of symbols are clearly a departure from the achievements by Stifel (1553). As to the superior way of solving problems with multiple unknowns let us look at the fifth problem which Gosselin solves by 'absolute quantities'. The problem text and its meta-description is as follows (Gosselin 1577, f. 82v):

> Quatuor numeros invenire, quorum primus cum semisse reliquorum faciat 17. Secundus cum aliorum triente 12. Tertius cum aliorum quadrante 13. Quartus cum aliorum sextante faciat 13.

$$\begin{aligned} x + \tfrac{y}{2} + \tfrac{z}{3} + \tfrac{u}{2} &= 17 \\ \tfrac{x}{3} + y + \tfrac{z}{3} + \tfrac{u}{3} &= 12 \\ \tfrac{x}{4} + \tfrac{y}{4} + z + \tfrac{u}{4} &= 13 \\ \tfrac{x}{6} + \tfrac{y}{6} + \tfrac{z}{6} + u &= 13 \end{aligned}$$

This is the very same problem of Buteo (1559, 193-6) shown in Figure 2.6. Bosmans (1906, 64) writes that here "Gosselin triumphs over Buteo who gets confused in solving the problem". Let us first look at Gosselin's solution in Table 2.10.

23	$11z = 88$	subtract 120 from (22)	tollamus 12D hoc est 120, restabunt 88 aequalia 11C
24	$z = 8$	divide (23) by 11	dividemus 88 in 11, quotus erit 8, valor C et tertius numerus
25	$3y + 10 + 32 = 54$	substitute (21) and (24) in (14)	sed etiam 3B4C1D aequalia sunt 54,
26	$3y = 12$	subtract 42 from (25)	tollamus hinc 4C1D, hoc est 10 et 32, nempe 42, restabunt 12 aequalia 3B
27	$y = 4$	divide (26) by 3	estque B et secundus numerus 4
28	$2x + 4 + 8 + 10 = 34$	substitute (21), (24) and (27) in (5)	iam vero 2A1B1C1D aequantur 34,
29	$2x = 12$	subtract 22 from (28)	tollamus 1B, nempe 4, 1C 8, 1D 10, hoc est 22, restabunt 12 aequalia 2A
30	$x = 6$	divide (28) by 2	quare 1A et primus numerus est 6

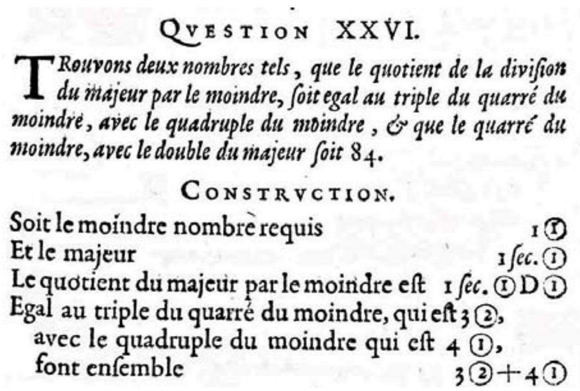
Table 2.11: Final part of Gosselin's solution to a problem from Buteo.

Buteo provides three different but correct solutions to the problem. In the first he reduces the number of equations by multiplication and subtraction to eliminate an unknown in every subtraction step. Gosselin's method may be somewhat more resourceful but there is little conceptual difference between both with regards to equations and the possible operations on equations. Remark that the solution text is close to identical with our meta-description.

This signifies the completion of an important phase towards the emergence of symbolic algebra.

2.11 Simon Stevin (1585)

In his *L'arithmetique*, Simon Stevin (1585) employs the second unknown for several problems. From questions 25 to 27 it becomes obvious that he used Cardano's *Ars Magna* for his use of the second unknown. Although not original in its method, Stevin's use of symbolism is quite novel (see Figure 2.7). Let us look at question 27 asking for three numbers in GP with the sum given and the condition that the square of the middle term is equal to twice the product of the two smaller numbers plus six times the smaller number (Stevin 1585, 402-404). In modern symbolism the structure of the problems is:



QVESTION XXVI.

TRouvons deux nombres tels, que le quotient de la diviſion du majeur par le moindre, ſoit egal au triple du quarré du moindre, avec le quadruple du moindre, & que le quarré du moindre, avec le double du majeur ſoit 84.

CONSTRVCTION.

Soit le moindre nombre requis 1 ①
Et le majeur 1 ſec. ①
Le quotient du majeur par le moindre eſt 1 ſec. ① D ①
Egal au triple du quarré du moindre, qui eſt 3 ②,
avec le quadruple du moindre qui eſt 4 ①,
font enſemble 3 ② + 4 ①

Fig. 2.7: Simon Stevin's symbolism for the second unknown (from Stevin 1585, 401)

$$a : b = b : c$$
$$a + b + c = d$$
$$eab + fa = b^2$$

Cardano discusses the problems with (20; 2, 4) for the values of (d; e, f). Stevin writes that he has the problem from Cardano and changes the values to (26; 2, 6). Stevin calls his solution a construction (of an equation) and starts by using the first unknown for the middle term and the second for the lower extreme, for which we will use x and y. An unknown is represented by

Stevin as a number within a circle. The number inside denotes the power of the unknown. Thus

$$① \text{ stands for } x \text{ and } ② \text{ for } x^2$$

To differentiate the second unknown from the first the power of the unknown is preceded by sec., for example

$$5y^2 \text{ becomes } 5sec.\ ②$$

For multiplication, Stevin used the letter M, thus

$$5xy^2 \text{ would be } 5\ ①M sec.\ ②$$

Remark that if this system would be extended to *pri.* and *ter.*, the circled numbers correspond to our exponents and the Stevin's symbolism becomes very similar with the one adopted by Descartes in 1637.

Stevin proceeds by formulating the condition in terms of the two unknowns as $x^2 = 2xy + 6y$, or using his notation, as

$$1② \text{ égale à } 2\ ①M sec.\ ① + 6sec.②$$

As x is the mean proportional between y and the third number c, $x^2 = yc$ and the larger extreme must be equal to $2x + 6$. Thus, y, x and $2x + 6$ are in continuous proportion and their sum is 26. This allows Stevin to express the value of the second unknown as:

$$-3① + 20$$

Substituting $(-3x + 20)$ as the value of y in $x^2 + 2xy + 6y$ leads to

$$x^2 = -6x^2 + 22x + 120$$

for which Stevin gives the root of 6 leading to the solution (2, 6, 18).

2.12 Conclusion

We have treated the development of symbolism with regards to the second unknown from 1539 to 1585, the period preceding Viète's *Isagoge* (1591). We have argued that the search – or we might even say, the struggle – towards a satisfactory system for representing multiple unknowns has lead to the creation of a new mathematical object: the symbolic equation. The solution to linear problems by means of the second unknown initiated, for the first time, operations on equations (in Cardano's *Practica Arithmeticae*) and operations between equations (in Cardano's *Ars Magna*). Once operations on equations became possible, the symbolic equation became a mathematical object of its own and hence required a new concept. Algebraic practice before Cardano consisted mostly of problem solving by means of the manipulation of polynomials – on the condition that they were kept equal – in order to arrive at a format for which a standard rule could be applied. We therefore use the term 'co-equal polynomials' for these structures rather than "equations" in the modern sense. Half a century of algebra textbooks marked the transition from algebra as a practice of problem solving (the abbaco and cossic tradition) to algebra as the study of equations. These authors, and especially Cardano and Stifel paved the Royal road for Viète, Harriot, and Descartes, to use algebra as an analytic tool within the wider context of mathematics. In order to study the structure of equations, the concept of a symbolic equation had to be established.

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