Chapter 5
The symbolic treatment of Euclid's *Elements* in Hérigone's *Cursus mathematicus* (1634, 1637, 1642)

Maria Rosa Massa Esteve

Abstract The publication in 1591 of *In artem analyticem isagoge* by François Viète (1540–1603) constituted an important step forward in the development of a symbolic language. This work was diffused through many other algebra texts, such as the section entitled *Algebra* in the *Cursus mathematicus, nova, brevi et clara methodo demonstratus, per notas reales & universales, citra usum cuiuscunque idiomatis, intellectu faciles* (Paris, 1634/1637/1642) by Pierre Hérigone (1580–1643). In fact, Hérigone's aim in his *Cursus* was to introduce a symbolic language as a universal language for dealing with both pure and mixed mathematics using new symbols, abbreviations and margin notes. In this article we focus on the symbolic treatment of Euclid's *Elements* in the first volume of the *Cursus* in which Hérigone replaced the rhetorical language of Euclid's *Elements* by symbolic language in an original way. Since Hérigone stated that he had followed Clavius's *Elements* (1589) in the writing of this first volume, we compare some demonstrations found in both authors' works as regards the style and the use of other propositions from Euclid's *Elements*, with the aim of clarifying the significance and the usefulness of Hérigone's new method of demonstration for a better understanding of mathematics.

Key words: Pierre Hérigone; Symbolic language; *Cursus mathematicus*; Euclid's *Elements*; Seventeenth century; Clavius's *Elements*.

Centre de Recerca per a la Història de la Tècnica, Departament de Matemàtica Aplicada I. Universitat Politècnica de Catalunya

5.1 Introduction

Pierre Hérigone[1] (1580–1643) published his *Cursus Mathematicus* (1634/1637/1642) in a period when the algebraization of mathematics was taking place. One of the fundamental characteristics of this process was the introduction of algebraic procedures to solve geometric problems.[2] In this process, the creation of a formal symbolic language to represent algebraic equations and geometric constructions and curves became one of algebra's essential features.[3] For this reason, the publication in 1591 of *In Artem Analyticen Isagoge* by François Viète (1540–1603) constituted an important step forward in the development of a symbolic language for mathematics.[4]

Viète's work was transmitted through various texts on algebra, such as the *Algebra* section of Hérigone's *Cursus Mathematicus* (hereafter referred to as the *Cursus*). We have analyzed this section in a recently published article (Massa, 2008), in which we show that while Hérigone used Viète's statements to deal with equations and their solutions, his notation, presentation and procedures were indeed quite different. Furthermore, we analyzed some of Hérigone's improvements that derived from a generalization of Viète's examples.

We now focus our research on the symbolic treatment of Euclid's *Elements* in the first volume of Hérigone's *Cursus* and its usefulness for rendering Math-

[1] Very little is known about Hérigone's life. Per Stromholm claims that he was from the Basque Country and that he taught mathematics in Paris. For more information see Stromholm (1972, p.6) and Knobloch (2001, p.13–14).

[2] Therefore, two new developments occurred in mathematics: first, the creation of what is now named analytic geometry, and second, the emergence of infinitesimal calculus. The two new disciplines achieve their ends through connections between algebraic expressions and geometric curves, on the one hand, and between algebraic operations and geometric constructions on the other. There are many useful studies on this subject, including Mahoney (1980, p.141–156), Mancosu (1996, p.84–86) and Panza (2005).

[3] In fact, the notation is not present in algebraic works in Arabic. Abbreviations are first used to represent the unknown quantities in the arithmetic works of the Renaissance period and algebraic procedures were expressed in syncopate form. The widespread use of symbolic notation began in the middle of the sixteenth century. There are many useful studies on the evolution of symbolic language, including Wallis (1685), Cajori (1928–29), Pycior (1997) and Stedall (2002).

[4] Viète used symbols to represent both known and unknown quantities, and was thus able to investigate polynomial equations in a completely general form. He conceived of equations in terms of Euclidean ideas of proportion. The equation $x^2 + bx = d^2$, for example, can be written as $x(x + b) = d^2$ and therefore as a proportion $x : d = d : (x + b)$. Solving the equation is therefore equivalent to finding three lines in continued proportion. Viète showed the usefulness of algebraic procedures for analysing and solving problems in arithmetic, geometry and trigonometry. The purpose of Viète's analytical art, in his own words, was to solve all kinds of problems. For more information see Viète (1646), Giusti (1992) and Bos (2001).

ematics more comprehensive. The aim of this paper is to show how Hérigone replaces the rhetorical language of Euclid's *Elements* with a symbolic language, as well as to analyze some examples of this procedure as a useful means of obtaining new results.

Since Hérigone stated in the *Prolegomena* to his *Elements* that he had followed the *Elements* (1589) of Christoph Clavius (1538–1612) for the writing of this first volume, we compare some demonstrations found in the texts by both authors, examining their style and the order and use of other propositions from Euclid's *Elements* in order to clarify the significance and the usefulness of reformulating rhetorical text into symbolic language.

We divide the article into three sections: the first section deals with the features of Hérigone's "new method" in the *Cursus*, the second describes his procedure of symbolically treating Euclid's *Elements* to make demonstrations, and our final section analyzes some examples of geometrical propositions in Hérigone's *Elements*, which facilitated the production of new demonstrations in the *Cursus*.

5.2 Hérigone's new method

In order to understand the reasoning used by Hérigone in his work, we must analyze the principal features of Hérigone's new method of demonstration described in the *Cursus*: the original system of notation, the axiomatic-deductive reasoning and the presentation of the propositions.

Hérigone wrote an encyclopaedic textbook consisting of five volumes known as the *Cursus Mathematicus*.[5] The first four volumes were published in 1634. The first and second volumes of the *Cursus* deal with pure mathematics. The first volume deals with geometry and the second volume is devoted to arithmetic and algebra. The third and fourth volumes deal with mixed mathematics, that is to say, with the mathematics required for practical geometry, military or mechanical uses, geography, and navigation. The fifth and last volume of the first edition, published in 1637, includes spherical trigonometry and music. Later, in the second edition (1642), Hérigone added the sixth and final volume, which contains two parts dealing with algebra; it also deals with perspective and astronomy.

Published in parallel Latin and French columns on the same page, the first edition, whose full title is *Cursus mathematicus, nova, brevi et clara methodo*

[5] Hérigone published an edition of the first six books of Euclid in 1639 (Hérigone, 1639), but Stromholm (1972, p.299) claims that these are "little more than the French portion of Volume 1 of the *Cursus*." For more information on the parts of the *Cursus* see Massa (2008, p.287).

demonstratus, per notas reales & universales, citra usum cuiuscunque idiomatis, intellectu faciles ["Course of Mathematics demonstrated by a brief and clear new method through real and universal symbols,[6] which are easily understood without the use of any language"],[7] states that Hérigone devised a new method of demonstration to understand Mathematics in a straightforward manner.

Hérigone also claimed that he had invented a new method for making demonstrations briefer and more intelligible that did not require the use of any language. In the preface to the first volume, which bore the dedication "Au lecteur" [To the reader] he explains,

> There is no doubt at all that the best method for teaching the sciences is that in which brevity is combined with ease. But it is not always easy to attain both, particularly in mathematics, which, as Cicero pointed out, is highly obscure. Having considered this myself, and seeing that the greatest difficulties arise from an understanding of the demonstrations, on which the knowledge of all parts of mathematics depend, I have devised a new method, brief and clear, of making demonstrations, without the use of any language.[8]

Indeed, Hérigone's stated aim in the *Cursus* was to introduce a symbolic language as a universal language for dealing with both pure and mixed mathematics. Moreover, Hérigone stressed the importance of knowing the symbols and understanding the demonstrations performed with this notation. His way of reasoning through the steps of the demonstration is axiomatic-deductive, as we explain below.

Thus, the first feature of Hérigone's new method is his system of notation; he uses many new symbols and abbreviations (which he calls "notes") and

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[6] We have translated the expression "notes" as "symbols;" however, in Hérigone's view "notes" include symbols and abbreviations.

[7] The title in French is "Cours Mathematique demonstré d'une nouvelle briefve et Claire methode. Par notes reelles & universelles, qui peuvent estre entendues sans l'usage d'aucune langue." In writing this article the author has referred to the copy held in the Bibliothèque Nationale de France.

[8] Car on ne doute point, que la meilleure methode d'enseigner les sciences est celle, en laquelle la briefveté se trouve conjoincte avec la facilité : mais il n'est pas aisé de pouvoir obtenir l'une & l'autre, principalement aux Mathematiques, lesquelles comme temoigne Ciceron, sont grandement obscures. Ce que considerant en moy-mesme, & voyant que les plus grandes difficultez estoient aux demonstrations, de l'intelligence desquelles dépend la cognoissance de toutes les parties des Mathematiques : i'ay inventé une nouvelle methode de faire les demonstrations, briefve & intelligible sans l'usage d'aucune langue. /Nam extra controversiam est, optimam methodum tradendi scientias, esse eam, in qua brevitas perspicuitati coniungitur, sed utramque assequi hoc opus hic labor est, praesertim in Mathematicis disciplinis, quae teste Cicerone, in maxima versantur difficultate. Quae cum animo perpenderem, perspectumque haberem, difficultates quae in erudito Mathematicorum pulvere plus negotij facessunt, consistere in demonstrationibus, ex quarum intelligentia Mathematicarum disciplinarum omnis omnino pendet cognitio : excogitavi novam methodum

margin notes (which he calls "citations"). We may claim that his notation is entirely original; indeed, most of the symbols had not appeared in any previous book. For example, in *Algebra*, Hérigone, like Viète, uses vowels to represent unknown quantities and consonants to represent known or given quantities. To represent powers, Hérigone writes the exponents on the right side of the letter (so the square is represented by a 2, the cube by a 3 and so on). See table 5.1 below.

Signs	Viète (1590s)	Harriot (1631)	Hérigone (1634)	Descartes (1637)
Equality	*Aequalis*	$=$	$2 \mid 2$	$\propto$
Greater than	*Maior est*	$>$	$3 \mid 2$	Plus grande
Less than	*Minus est*	$<$	$2 \mid 3$	Plus petite
Product of a and b	A and B	ab	ab	ab
Addition	*plus*	$+$	$+$	$+$
Subtraction	*minus*	$-$	$\sim$	$-$
Ratio	*ad*		$\prod$	à
Square root	$VQ.$	$\sqrt{\ }$	$V2$	$\sqrt{\ }$
Cubic root	$VC.$	$\sqrt{c}$	$V3$	$\sqrt{c}$
Squares	*Aquadratus, Aquad*	aa	$a2$	a^2, aa
Cubes	*Acubus, Acub*	aaa	$a3$	a^3

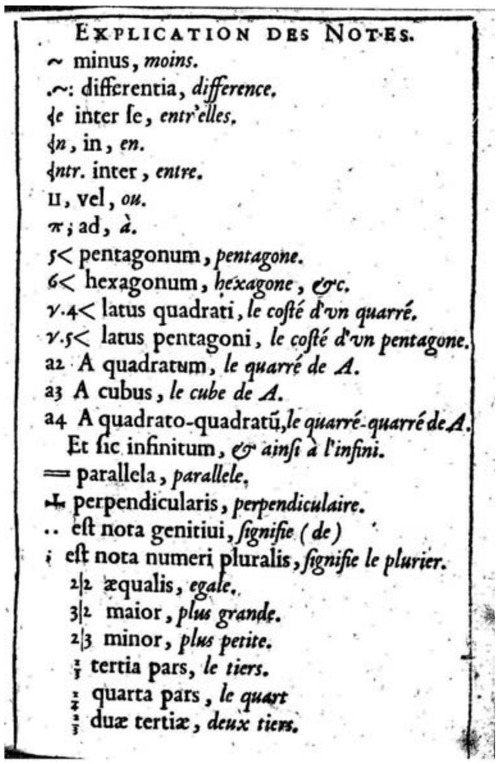
Table 5.1: Table of notations from Massa (2008, p.289).

Furthermore, Hérigone provides alphabetically ordered explanatory tables of abbreviations and symbols (which he calls "explicatio notarum"). For example, there is a mark for the side of the square, a sign meaning 'perpendicular', and a symbol for representing ratios. (See figure 5.1.)

Hérigone also gives explanatory tables for the citations (which he calls "explicatio citationum") at the beginning of each of the volumes of which the *Cursus* is composed. The citations always refer either to propositions in Euclid's *Elements* or to the *Cursus* itself. In the margin of the demonstrations of propositions, Hérigone cites, line by line, the numbers corresponding to the theorems he has used.[9]

demonstrandi brevem & citra ullius idiomatis usum intellectu facilem. (Hérigone, 1634, I, *Ad Lectorem*). All translations are the author's own.

[9] In the Ancient copies of Greek editions of Euclid there are no references in the margin to the theorems he used. However, these references are introduced in Renaissance editions of Euclid, particularly in Clavius, which was evidently Hérigone's model, as he himself points out. We would like to draw attention to Hérigone's elucidation of Clavius, in which it is not just Clavius's works that are mentioned; Hérigone explained that he had used Clavius's order and text for Euclid's *Elements*, as well as for the three books of Theodosius's *Spherics* and for the fourth book up to the eighteenth proposition. See Hérigone (1642, VI, p. 241).


EXPLICATION DES NOTES.

~ minus, *moins.*
.~: differentia, *difference.*
ɫe inter ſe, *entr'elles.*
ɫn, in, *en.*
ɫntr. inter, *entre.*
u, vel, *ou.*
π; ad, *à.*
5< pentagonum, *pentagone.*
6< hexagonum, *hexagone, &c.*
γ.4< latus quadrati, *le coſté d'vn quarré.*
γ.5< latus pentagoni, *le coſté d'vn pentagone.*
a2 A quadratum, *le quarré de A.*
a3 A cubus, *le cube de A.*
a4 A quadrato-quadratũ, *le quarré-quarré de A.*
Et ſic infinitum, *& ainſi à l'infini.*
= parallela, *parallele.*
⊥ perpendicularis, *perpendiculaire.*
.. eſt nota genitiui, *ſignifie (de)*
; eſt nota numeri pluralis, *ſignifie le plurier.*
2|2 æqualis, *egale.*
3|2 maior, *plus grande.*
2|3 minor, *plus petite.*
1/3 tertia pars, *le tiers.*
1/4 quarta pars, *le quart*
2/3 duæ tertiæ, *deux tiers.*

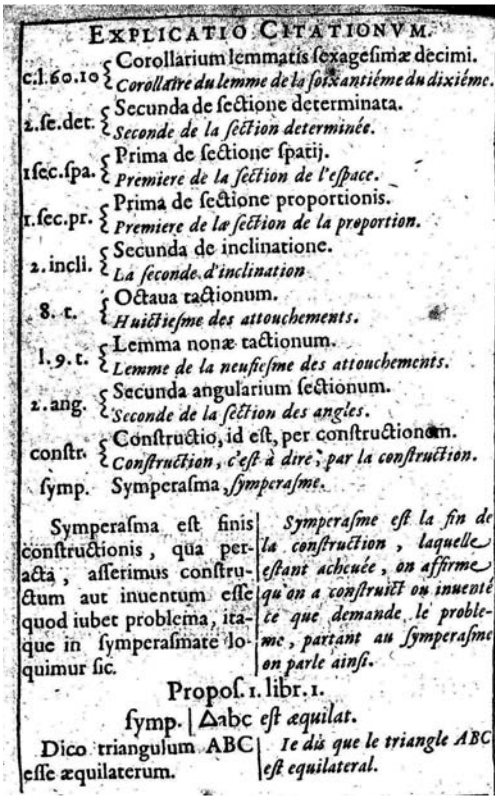
Fig. 5.1: Hérigone's table of abbreviations (Hérigone, 1634, I, f. bv^r)

Thus, for example, "c.l.60.10" means "Corollary of the lemma of the proposition X.60" (See figure 5.2).[10]

The second feature of Hérigone's method is the axiomatic-deductive reasoning explicitly described by him. In the preface to the reader, Hérigone emphasizes that the introduction of margin notes is key for following the steps of the demonstration and this trait is used in this method, unlike in the "vulgar and common" or ordinary method. He criticizes other authors who use the "vulgar and common" method. We do not know the exact meaning of this expression, but since it was Hérigone's belief that it was difficult to understand the demonstrations, this expression acquires its significance for

On Clavius and his influence on other seventeenth-century authors, see Knobloch (1988) and Rommevaux (2006).

[10] Corollaire du lemme de la soixantième du dixième./ Corollarium lemmatis sexagesimae decimi (Hérigone, 1634, I, unpaginated).



EXPLICATIO CITATIONVM.

c.l.60.10	Corollarium lemmatis sexagesimæ decimi. / *Corollaire du lemme de la soixantiéme du dixiéme.*
2.se.det.	Secunda de sectione determinata. / *Seconde de la section determinée.*
1sec.spa.	Prima de sectione spatij. / *Premiere de la section de l'espace.*
1.sec.pr.	Prima de sectione proportionis. / *Premiere de la section de la proportion.*
2.incli.	Secunda de inclinatione. / *La seconde d'inclination*
8.t.	Octaua tactionum. / *Huictiesme des attouchements.*
l.9.t.	Lemma nonæ tactionum. / *Lemme de la neufiesme des attouchements.*
2.ang.	Secunda angularium sectionum. / *Seconde de la section des angles.*
constr.	Constructio, id est, per constructionem. / *Construction, c'est à dire, par la construction.*
symp.	Symperasma, *symperasme.*

Symperasma est finis constructionis, qua peracta, asserimus constructum aut inuentum esse quod iubet problema, itaque in symperasmate loquimur sic.	*Symperasme est la fin de la construction, laquelle estant acheuée, on affirme qu'on a construict ou inuenté ce que demande le probleme, partant au symperasme on parle ainsi.*

Propos. 1. libr. 1.

symp. | △abc *est æquilat.*

Dico triangulum ABC esse æquilaterum.	*Ie dis que le triangle ABC est equilateral.*

Fig. 5.2: Hérigone's explanatory table of citations (Hérigone, 1634, I, f. bvii^v)

designing the methods used by other authors in contrast to the new method he is introducing. In Hérigone's own words:

> I also stress that in the ordinary method many words and axioms are used without prior explanation, but in this method there is nothing that has not already been explained and conceded in the premises; even in the demonstrations, which are somewhat longer, all that was proved in the sequence of the demonstration are cited with Greek letters.[11]

[11] Soient aussi qu'en la methode ordinaire on se sert beaucoup de mots & d'axiomes sans les avoir premierement expliquez, mais en cette methode on ne dit rien qui n'aye esté expliqué & concedé aux premises ; mesme aux demonstrations, qui sont quelque peu longues, on cite par lettres Grecques, ce qui a esté demonstrée en la suite de la démonstration. /Huc etiam accedit, quòd in vulgari & communi docendi ratione, plurima proferantur vocabula,& axiomata absque ulla illorum in praemisis explicatione : sed in hac methodo nihil adfertur, nisi fuerit in praemissis explicatum & concessum. Quum etiam longiores occurrunt demonstra-

Hérigone goes on to describe his axiomatic-deductive reasoning for the demonstrations, and adds that he will give an example in the first proposition of the first book. In Hérigone's own words,

> And as each consequence depends immediately on the proposition cited, the demonstration follows from beginning to end by a continue series of legitimate, necessary and immediate consequences, each one included in a short line, which can be solved easily by syllogisms, because in the proposition cited as well as in that which corresponds to the citation one can find all parts of the syllogism, as one may see in the first demonstration of first book, which has been reduced by syllogisms.[12]

Hérigone's originality resides not only in the explicit explanation of axiomatic-deductive reasoning, but also because one can find in one symbolic line the major premise and the conclusion, using the former symbolic line as the minor premise. In the following section we analyse the syllogism and the identification of the premises in the demonstration.

The third feature of Hérigone's method of demonstration is the presentation of propositions. He also stresses this point in the preface to the reader,

> The distinction of the proposition in its members, that is, the part in which the hypothesis is advanced, the explanation of the requirement, the construction or preparation and the demonstration, likewise relieves the memory and makes it very helpful for understanding the demonstration.[13]

Indeed, Hérigone's propositions are proved from hypotheses and well-established properties. Sometimes he states the equalities that he needs for the demonstration in a "Praeparatio" paragraph after the hypothesis. He also divides his demonstrations into separate sections: hypothesis (known and unknown

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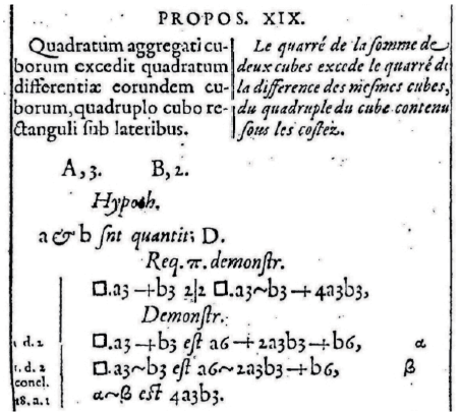
tiones, quae iam in serie demonstrationis sunt probata, litteris Graecis citantur (Hérigone, 1634, I, *Ad Lectorem*).

[12] Et parce que chaque consequence depend immediatement de la proposition citée, la demonstration s'entretien depuis son commencement jusques à la conclusion, par une suite continue de consequences legitimes, necessaires & immediates, contenues chacune en une petite ligne, lesquelles se peuvent resoudre facilement en syllogismes, à cause qu'en la proposition citée, & en celle qui correspond à la citation, se trouvent toutes les parties du syllogisme: comme on peut voir en la premiere demonstration du premier livre, qui a esté reduite en syllogismes. /Et quoniam singulae consequentiae ex propositionibus allegatis immediate pendent, demonstratio ab initio ad finem, serie continua, legitimarum, necessariarumque consecutionum immediatarum, singulis lineolis comprensarum aptè cohaeret: quarum unaquaeque nullo negotio in syllogismum potest converti, quòd in propositione citata, & in ea quae citationi respondet, omnes syllogismi partes reperiatur: ut videre est in prima libri primi demonstratione, quae in syllogismos est conversa (Hérigone, 1634, I, *Ad Lectorem*).

[13] La distinction de la proposition en ses membres, savoir en l'hypothese, l'explication du requis, la construction, ou preparation, & la demonstration, soulage aussi la memoire, & sert grandement à l'intelligence de la demonstration. /Praeterea distinctio propositionis in sua membra, scilicet in hypothesin, explicationem quaesiti, constructionem, vel praeparationem, & demonstrationem non parum iuvat quoque memoriam, & ad intelligendam demonstrationem multùm prodest. (Hérigone, 1634, I, *Ad Lectorem*).

quantities); explanation or requirement; demonstration, and conclusion. In the margin he writes the number of propositions of Euclid's *Elements* that he is using. He occasionally gives the numerical solution (for example in an equation) in a section headed "Determinatio". In geometric constructions, he provides the instructions needed to make the drawing in a paragraph referred to as "Constructio".[14]

Let us see how Hérigone works when proving an algebraic identity in the *Algebra* (see Figure 5.3). He proves the algebraic identity, which in modern


PROPOS. XIX.

Quadratum aggregati cuborum excedit quadratum differentiæ eorundem cuborum, quadruplo cubo rectanguli sub lateribus. | Le quarré de la somme de deux cubes excede le quarré de la difference des mesmes cubes, du quadruple du cube contenu sous les costez.

A, 3. B, 2.

Hypoth.

a & b *snt quantit*: D.

Req. π. demonstr.

□.a3 + b3 2|2 □.a3 ~ b3 + 4a3b3,

Demonstr.

1 d. 2 — □.a3 + b3 *est* a6 + 2a3b3 + b6, α

1. d. 2 — □.a3 ~ b3 *est* a6 ~ 2a3b3 + b6, β

concl. 18. a. 1 — α ~ β *est* 4a3b3.

Fig. 5.3: Proposition XIX in *Algebra*'s chapter 5. (Hérigone, 1634, II, p. 46) Reproduced from the BNF microfilm.

notation would be expressed $(a^3 + b^3)^2 = (a^3 - b^3)^2 + 4a^3b^3$, as follows:

> The square of the sum of two cubes exceeds the square of the difference of the same cubes by the quadruple of the cube determined by the sides.[15]

[14] We would also like to point out that Pietro Mengoli (1626-1686), Hérigone's follower, writes all his demonstrations in Hérigonean style by dividing them into a "Hypothesis," "Demonstratio," "Praeparatio" and "Constructio." Furthermore, in the margin he cites line by line all the propositions and properties he has used according to an axiomatic-deductive reasoning. Thus, under the influence of Hérigone, who considered Euclid's *Elements* the point of reference par excellence, Mengoli brings together, as he says, a "conjuntis perfectionibus" [perfect conjunction] of classical mathematics and modern mathematics to obtain new theories and new results. See Massa (1997, 2003, 2006a, 2006b, 2009).

[15] Le quarré de la somme de deux cubes excede le quarré de la difference des mesmes cubes, du quadruple du cube contenu sous les costez. /Quadratum aggregati cuborum excedit quadratum differentiae eorundem cuborum, quadruplo cubo rectanguli sub lateribus (Hérigone, 1634, II, p. 46).

Hérigone's Notation	Modern Notation
Hypoth.	Hypothesis
a & b *snt quantit*; *D.*	a and b are given quantities.
Req. II. Demonstr.	It is required to prove that:
$.\square a3 + b32\|2.\square a3 \sim b3 + 4a3b3$, *Demonstr.*	$(a^3 + b^3)^2 = (a^3 - b^3) + 4a^3b^3$, Demonstration.
$1.d.2^{16}.\square.a3 + b3$ *est* $a6 + 2a3b3 + b6, \alpha$	II.def.1 $(a^3 + b^3)^2$ is $a^6 + 2a^3b^3 + b^6, (\alpha)$
$1.d.2.\square.a3 \sim b3$ *est* $a6 \sim 2a3b3 + b6, \beta$	II.def.1 $(a^3 - b^3)^2$ is $a^6 - 2a^3b^3 + b^6, (\beta)$
Concl. $18.a.1.^{17}$ $\alpha \sim \beta$ *est* $4a3b3$.	Conclusion. I. axiom.18 $\alpha - \beta$ is $4a^3b^3$.

Table 5.2: Modern translations of Hérigone's notations

It is worth pointing out that Hérigone formulates the identity to prove and even the definitions and axiom used in symbols, without rhetorical explanations or verbal descriptions. He also divides his demonstration into separate sections: Hypothesis, requirement to prove, demonstration and conclusion.

We may conclude that Hérigone was convinced that this new method of demonstration with his new system of notation, his axiomatic-deductive reasoning and his new manner of presentation is the clearest, most concise and most suitable for rendering the mathematics more comprehensively. In the preface, after analyzing the features of his new method Hérigone affirms: "These are the principal commodities to be found in our new method of demonstration".[18]

5.3 The reformulation of Euclid's *Elements* in symbolic language

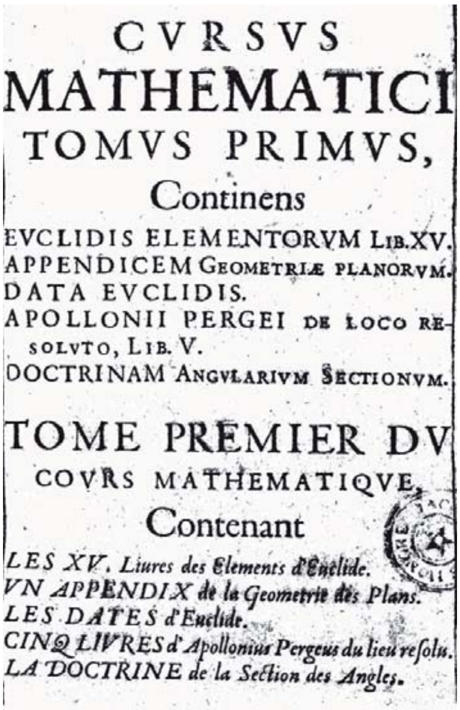
The first volume of the *Cursus* contains Euclid's *Elements* and *Data*, Apollonius's *Conics*[19] and an exposition of Viète's *Doctrine of angular sections* (see Figure 5.4). Hérigone presents the fifteen[20] books of Euclid's *Elements*, which is also one of the first translations of Euclid's *Elements* into a symbolic language. In fact, Isaac Barrow (1630–1677) in the letter *Ad lectorem* in his own edition of the *Elements* (1659), mentioned Hérigone as an example to

[18] Voila les principales commoditez qui se trouvent en notre nouvelle méthode de demonstrer. /Atque haec sunt commoda, quae in hac nova methodo demonstrandi reperiuntur (Hérigone, 1634, I, unpaginated).

[19] At the end of Euclid's *Data*, Hérigone's stated aim was to introduce his new method of demonstration into the five texts on Apollonius's *Conics* restored by Snell (3 texts), Ghetaldi and Viète as well as into the section of angles invented by Viète. (Hérigone 1634, I, p.889–935).

[20] Hérigone, like Clavius, mentions that only the first thirteen books are attributed to Euclid and that the other two are attributed to Hypsicles Alexandrinus (Hérigone, 1634, I,

follow both for reducing Euclid's *Elements* to one volume and for turning it into a symbolic language (Barrow, 1659, unpaginated).[21]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EVCLIDIS ELEMENTORVM LIB. XV.
APPENDICEM GEOMETRIÆ PLANORVM.
DATA EVCLIDIS.
APOLLONII PERGEI DE LOCO RE-
SOLVTO, LIB. V.
DOCTRINAM ANGVLARIVM SECTIONVM.
TOME PREMIER DV
COVRS MATHEMATIQVE,
Contenant
LES XV. Liures des Elements d'Euclide.
VN APPENDIX de la Geometrie des Plans.
LES DATES d'Euclide.
CINQ LIVRES d'Apollonius Pergeus du lieu resolu.
LA DOCTRINE de la Section des Angles.

Fig. 5.4: Hérigone's frontispiece to Volume I of the *Cursus*. (Hérigone, 1634, I, f. aii^{v}).

Although Hérigone uses the Latin version of Clavius's 1589 edition of the *Elements* only the statements and some figures for the propositions match

Prolegomena).

[21] Barrow for his part explained that Hérigone's reformulation is for the gratification of those readers who prefer symbolical to verbal reasoning. In his introduction, Heath also explained this circumstance when he described the principal translations and editions of the Elements. "The first six books 'demonstrated by symbols, by a method very brief and intelligible' by Pierre Hérigone, mentioned by Barrow as the only editor before him who had used symbols for the exposition of Euclid" (Heath, 1956, p.108). However, Barrow was partially mistaken, since Oughtred, in 1631, in the first edition of the *Clavis Mathematicae* had also rewritten some propositions of Euclid's *Elements* in symbolic language. Harriot had also done this even earlier but his version was never published and remains in manuscript form. See Stedall (2007, p.386). On the influence of Hérigone's *Cursus*, see Cifoletti (1990)

in both texts. The style of Hérigone's propositions, dividing his demonstrations into separate sections, is not found in Clavius's *Elements*. Moreover, Clavius, unlike Hérigone, describes the demonstration and the corresponding construction for each of his propositions and problems rhetorically.

Like Clavius in his *Prolegomena*, Hérigone's *Prolegomena* to the *Elements* discusses the classification of Mathematics; however, Hérigone did not follow Clavius's classifications. In Clavius' *Prolegomena* the order of the parts (Arithmetic, Music, Geometry and Astronomy) and the division into pure and mixed mathematics are the same as those by Proclus in his commentary.[22] In contrast, Hérigone ordered the four parts as Arithmetic, Geometry, Astronomy and Music and while like Clavius he considered mathematics to be divided into pure and mixed mathematics, Hérigone only mentioned Optic, Mechanics, Astronomy and Music as mixed.[23]

Hérigone, in accordance with Clavius, divides the fifteen books of Euclid's *Elements* into four parts[24] and this paragraph in both *Prolegomena* is identical word for word. There is a further part in Hérigone's *Prolegomena* called "The principles of Mathematics,"[25] which is also very similar to the corresponding part in Clavius. Both considered the principles of Mathematics as being divided into three types: the definitions, the postulates and the axioms or common notions.[26]

However, Hérigone goes further to add new "scholia" to Clavius's propositions, which he later uses to justify his demonstrations, and an appendix to

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and Massa (2008, p.298–299).

[22] Hérigone explained that the Pythagoreans divided mathematics into four categories: arithmetic, geometry, astronomy and music. He said others divided mathematics into pure and mixed mathematics, specifying that in pure mathematics quantity was recognized as being separate from matter. He considered that pure mathematics should be divided according to the kind of quantity (either continuous or discrete) into geometry and arithmetic, and that mixed mathematics should be divided into optics, mechanics, astronomy and music. See Hérigone (1634, I, *Prolegomena*). Clavius also divided mathematics into pure and mixed Mathematics, pure Mathematics includes Arithmetic and Geometry and mixed Mathematics includes Astrology, Perspective, Geodesy, Canonical or Music, Calculation and Mechanics. See Clavius (1589, section II, *Prolegomena*).

[23] On the status of the mathematical disciplines in sixteenth century, see Axworthy (2004, p.62–80).

[24] The first part contains the first six books, which deal with planes. The second includes the subsequent three books, which deal with numbers. The third part contains only Book X, which deals with commensurable and incommensurable lines, while the last part is composed of the last five books, which treat the science of solids. See Hérigone (1634, I, *Prolegomena*). Like Clavius, Hérigone specifies the part corresponding to each book in the titles, for example, Book XI reads "The first book on the science of solids."

[25] Des principes des Mathematiques. /De principiis Mathematicis (Hérigone, 1634, I, *Prolegomena*).

[26] Hérigone claims that he added new axioms to the principles of Mathematics whenever he considered them necessary for the demonstrations. He specifies that he included a letter

Book VI, where Hérigone explains sums and products of lines, justifying them by propositions from his own *Elements*. In addition, Hérigone introduces this appendix with the claim that its problems and theorems are necessary for understanding Algebra and Astronomy.[27]

In fact, throughout the *Cursus*, Hérigone insists on the fundamental role of Euclid's *Elements* for understanding mathematics. Hérigone deals with geometry and arithmetic in the first and second volumes, respectively, and in the preface to the second volume he justifies treating geometry before arithmetic by claiming that geometry enables a better understanding of arithmetic:

> On the one hand, it is certain that knowledge of numbers is absolutely necessary for considering symmetry and incommensurability of a continuous quantity, of which Geometry constitutes one of the principal objects. On the other hand, there are some demonstrations in our arithmetic that cannot be understood without the help of the first books of Euclid's *Elements*.[28]

Moreover, when Hérigone discusses the importance of algebra in Volume VI (1642), he again stresses that the only requirement for solving the equations is an understanding of Euclid's *Elements*.[29]

We may assume that Hérigone believed that an understanding of Euclid's *Elements* also served a propaedeutic function in his *Cursus*.[30]

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to distinguish his new axioms from Clavius's and Euclid's axioms.

[27] A ces six livres des Elements d'Euclide, j'adiousteray un appendix de divers problèmes & theoremes, dont les uns sont necessaires à l'Algebre, les autres à l'Astronomie ; /His sex elementorum Euclidis libris, annectam variorum problematum atque theorematum appendicem; quorum alia ad Algebram, alia ad Astronomiam. [To these six books of Euclid's *Elements*, I add an appendix with some problems and theorems, some of which are necessary for Algebra and others for Astronomy.] (Hérigone, 1634, I, p.302).

[28] Car d'un coté il est constant que la connaissance des nombres est absolument requise à la considération de la symétrie et incommensurabilité de la quantité continue, desquelles la Géométrie fait un de ses principaux objets ; et d'autre part, il y a des démonstrations en notre Arithmétique qui ne peuvent être entendues sans le secours des premiers livres des Eléments d'Euclide. /Quantitatis enim continuae symmetriam & incommensurabilitatem, quas praecipue inquirit Geometra nusquam intelliget imparatus à numeris : Neque ex adverso percipi possunt Aritmeticae nostrae quaedam demonstrationes, sine previa cognitione priorum elementorum Euclidis. (Hérigone, 1634, II, unpaginated)

[29] Supplément de l'Algèbre . Les équations d'Algèbre sont d'autant plus difficiles à expliquer qu'elles sont hautes en l'ordre de l'échelle. Et n'est pas besoin d'autres préceptes particuliers, que de l'intelligence des éléments d'Euclide pour trouver la valeur d'une racine constituée en sa base. /Omnis algebrae aequatio quo altiorem scalae tenet locum, eo difficiliorem habet explicationem. Nec ullo praecepto particulari, praeter Euclidis elementorum notitiam, opus est, ad exhibendum radicis in sua base existentis valorem. [Supplement on Algebra. The higher the degree of equations in algebra, the more difficult it is to solve them. There is no need for particular rules other than an understanding of Euclid's *Elements* to find the value of a root that constitutes the base [of the equation]]. (Hérigone, 1642, VI, p.1)

[30] On the propaedeutic function in Euclid's *Elements*, Tartaglia and Clavius, see Axworthy (2004, p.13–38).

Volume I of the *Cursus* includes a translation into French of Euclid's *Elements*, which Hérigone reformulates in his new symbolic language in an original way. So all Euclidean propositions are expressed using symbolic expressions; for example, Pythagoras's theorem in Proposition I.47 from Euclid's *Elements* is expressed as "$\square.bc\ 2|2\ \square.ab + \square.ac.$"[31]

However, it is of the utmost importance to analyze how Hérigone replaces rhetorical language in the *Cursus* using his own *Elements* expressed in symbolic language. He introduces original symbols and abbreviations ("notes") and margin notes ("citations") to represent axioms, postulates and definitions. In fact, Hérigone classifies the citations used in the demonstrations as follows:

> There are seven types of citations in mathematical demonstrations, that is to say, the postulates, the problems, the definitions, the axioms, the theorems, the hypotheses and the constructions: of which the two first pertain to the construction or to the preparation and the other five to the demonstrations.[32]

His procedure for the citations is as follows: first, he writes the statement of the axiom, postulate or definition in rhetorical language similar to Clavius's *Elements*; second, he writes the symbol or abbreviation deduced from this axiom, postulate or definition, and finally, he offers an explanation of this abbreviation (*Explicatio notarum*). For example, the note "3.p.1." refers to Euclid's Postulate I. 3: "To describe a circle with any centre and distance" (see figure 5.5). Then Hérigone replaces Clavius's rhetorical language by these symbolic expressions and abbreviations defined previously. For example, where Clavius has "Centro *A*, & intervalo rectae *AB*, describatur circulus *CBD*," Hérigone writes "*abcd* est O" and notes in the margin "3.p.1.," referring to the sentence deduced from Euclid's Postulate I.3. Similarly, throughout Clavius's text Hérigone replaces rhetorical explanations by symbolic language. Let us take one example, the first proposition in Book I, where Hérigone uses this abbreviation and other similar ones in the construction and in the demonstration (see Figure 5.6). Hérigone's statement is expressed as follows: "On a finished straight line, to make an equilateral triangle."[33]

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[31] In these demonstrations, Hérigone writes a paragraph headed "praeparatio" in which he expresses parallel lines using the symbol "==," angles using the symbol "<" and a right angle using the symbol "□" (Hérigone, 1634, I, p.55–56).

[32] Aux demonstrations Mathematiques il y a sept genres de citations, à savoir, les postulats, les problèmes, les definitions, les axiomes, les theoremes, les hypotheses,& les constructions : desquels les deux premiers appartiennent à la construction, ou preparation, & les cinq autres à la demonstration. /In demonstrationibus Mathematicis sunt septem citationum genera, scilicet, postulata, problemata, definitiones, axiomata, theoremata, hypotheses, & constructiones: quorum duo priora, ad constructionem, aut praeparationem, reliqua quinque ad demonstrationem pertinent (Hérigone, 1634, I, Rrr iiij). This clarification is found at the end of volume 1 under the title: "Annotations on the first volume of *Cursus Mathematicus*".

[33] Sur une ligne droite donnée & terminée, descrire un triangle equilateral. /Super data

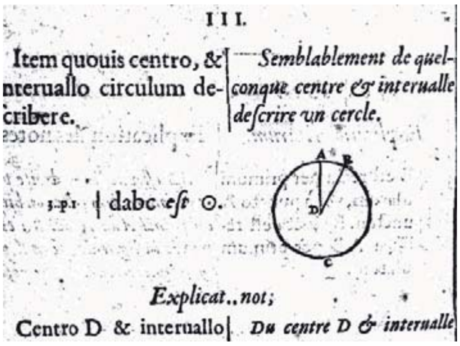

Fig. 5.5: Postulate I.3 of Hérigone's *Elements* (Hérigone, 1634, I, f. dvii^{v}).

In this proposition, both describe how to construct an equilateral triangle, Clavius using rhetorical explanations and Hérigone using his symbolic language with repeated references to Euclid's *Elements*. The connecting thread is the geometric construction of the solution. Hérigone replaces Clavius's rhetorical explanations and instructions with symbolic language. He proceeds by replacing each of Clavius's rhetorical sentences by his own corresponding abbreviation, and in the margin he makes a note referring to Hérigone-Euclid's propositions, postulates or axioms used and defined previously. For instance, when Clavius has, "Ex quarum utrovis, nempe ex *C*, ducantur duae rectae lineae *CA*, *CB*, ad puncta *A* & *B*," Hérigone writes the abbreviations "*ac & bc, snt* —" and makes a note in the margin "1.p.1.," thus referring to Postulate 1 of Book I of Hérigone's first volume. Similarly, for the demonstration, where Clavius has: "Quoniam rectae *AB*, *AC*, ducuntur ex centro *A*, ad circumferentiam circuli *CBD*, erit recta *AC*, recta *AB*, aequalis," Hérigone writes, "*ac* 2|2 *ab*" and makes a note in the margin "15.d.1.," referring to Euclid I. definition 15.

Like Clavius, Hérigone makes a new demonstration by syllogisms; however, the procedure is not exactly the same. Clavius makes the demonstration in a scholium and begins the sequence by the last syllogism of the demonstration. See for example, the order in Clavius's demonstration; he begins,

> All triangles that have three equal sides[35] are equilateral.
> The triangle *ABC* has three equal sides.
> Therefore the triangle *ABC* is equilateral.

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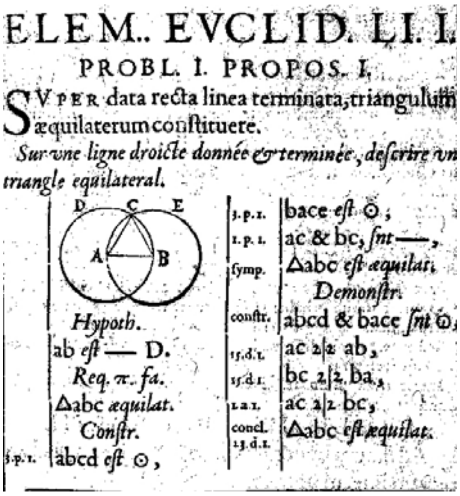
recta linea terminata, triangulum aequilaterum constituere. (Hérigone, 1634, I, p.1). The statement and the figure are identical to those of Clavius.

[35] Here Clavius makes a small letter "d" and in the margin he writes "d. 23.def." (Clavius, 1589, p.28).



ELEM.. EVCLID. LI. I.

PROBL. I. PROPOS. I.

SVPER data recta linea terminata, triangulum æquilaterum constituere.

Sur vne ligne droicte donnée & terminée, descrire vn triangle equilateral.

Hypoth.
ab *est* — D.
Req. π. fa.
Δabc *æquilat.*
Constr.
3.p.1. abcd *est* ⊙,
3.p.1. bace *est* ⊙;
1.p.1. ac & bc, *snt* —,
symp. Δabc *est aequilat.*
Demonstr.
constr. abcd & bace *snt* ⊙,
15.d.1. ac 2|2 ab,
15.d.1. bc 2|2 ba,
1.a.1. ac 2|2 bc,
concl. 23.d.1. Δabc *est æquilat.*

Hérigone's Notation	Modern Notation
Hypoth.	Hypothesis.
ab est—D	AB is a given straight line.
Req. π. *fa.*	It is required to make:
Δabc æquilat.	ABC equilateral triangle.
Constr.	Construction.
3.*p*.1. *abcd est* ⊙,	I.postulate.3. $ABCD$ is a circle of center A and distance AB,
3.*p*.1. *bace est* ⊙,	I.postulate.3. $BACE$ is a circle of center B and distance BA,
1.*p*.1. *ac* & *bc*, *snt*—,	I.postulate.1 AC and BC are straight lines,
Symp.[34] *Δabc est aequilat.*	Symperasma. I say that the triangle ABC is equilateral
Demonstr.	Demonstration.
Constr. abcd & *bace snt* ⊙,	Construction. $ABCD$ and $BACE$ are circles,
15.*d*.1. *ac* 2|2 *ab*,	I.definition.15. $AC = AB$,
15.*d*.1. *bc* 2|2 *ba*,	I.definition.15. $BC = BA$,
1.*a*.1. *ac* 2|2 *bc*,	I.axiom.1. $AC = BC$,
Concl.	Conclusion.
23.*d*.1. *Δabc est aequilat.*	I.definition.23. ABC is an equilateral triangle.

Fig. 5.6: Proposition I.1 (Hérigone, 1634, I, p.1) and modern translations of Hérigone's notations.

> The minor will be confirmed by this other syllogism:[36]

Clavius continues the demonstration by syllogisms until the first sentence, which is the construction of the circumference.

Hérigone also makes the same demonstration in a scholium, but explains the four syllogisms beginning with the first line of demonstration[37], and the major and minor premise as well as the conclusion can easily be identified in each syllogism. See Hérigone's demonstration by syllogisms:

> This demonstration is made by four syllogisms, as one can perceive from the number of citations.
> I SYLLOGISM.
> The straight lines traced from the centre to the circumference are equal to each other.
> But the straight lines AC & AB are traced from the centre to the circumference.
> Therefore the straight lines AC & AB are equal to each other.[38]

If we consider the citation: "I. definition. 15. $AC = AB$," we can see that the major premise is "I. definition. 15.," and that the minor premise is the line immediately preceding it: "$ABCD$ and $BACE$ are circles," and that the conclusion is: "$AC = AB$." For the second syllogism, Hérigone explains that it is the same as the first. The conclusions of the two first syllogisms serve for the minor premise in the third syllogism. Let us now consider the third syllogism:

> III SYLLOGISM.
> Things those are equal to the same are equal to each other.

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[36] Omne triangulum habent tria latera aequalia, est equilaterum. Triangulum ABC, tria habet aequalia latera. Triangulum igitur ABC, est aequilaterum. Minorem confirmabit hoc alio syllogismo. (Clavius, 1589, p.28). The same sequence by syllogisms is found in the appendix by Alessandro Piccolomini entitled *Commentarium de Certitudine Mathematicarum Disciplinarum* (Roma, 1547). (Piccolomini, 1547, p.99 r–99v). According to Rommevaux (2005, p.52), Clavius makes no reference to Piccolomini, although he probably knew this work, which forms part of the debate on the certainty of mathematics in the sixteenth century. There are many useful works on this *quaestio*, including Mancosu (1996) and Romano (2007).

[37] This same order of syllogisms is found in the demonstration by syllogisms of Dasypodio's work on Euclid's *Elements*. (Dasypodio, 1566, A ij).

[38] Cette demonstration se fait par quatre syllogismes, comme il appert du nombre des citations. I. SYLLOGISME. Les lignes droites menées du centre à la circonference, sont égales entre elles. Mais les lignes droites AC & AB sont menées du centre à la circonference. Donc les lignes droites AC & AB sont égales entr'elles. /Haec demonstratio sit quatuor syllogismis, ut perspicuum est ex numero citationum. I SYLLOGISMUS. Rectae lineae quae ducuntur à centro ad circunferentiam, sunt inter se aequales. Sed rectae AC & AB ducuntur à centro ad circunferentiam. Igitur rectae AC & AB sunt inter se aequales. (Hérigone, 1634, p.1–2)

But the straight lines AC & CB are equal to the same straight line.
Therefore the straight lines AC & CB are equal to each other.[39]

In this case, "I. axiom. 1.$AC = BC$," the major premise is the first axiom, while the minor premise is deduced from the conclusions of the first and second syllogisms: $AC = AB$ and $BC = BA$, and the conclusion of the third syllogism is $AC = BC$. These conclusions enable the minor premise in the last syllogism to be deduced.

IV SYLLOGISM.
All triangles that have three equal sides are equilateral.
But the triangle ABC has three equal sides.
Therefore the triangle ABC is equilateral.[40]

In this case, "I. definition. 23. ABC is an equilateral triangle," the major premise is I.d.23, while the minor premise is deduced from the former conclusions $AC = AB$, $BC = BA$ and $AC = BC$, and the conclusion of the third syllogism is that "the triangle ABC is equilateral," which concludes the demonstration.

Hérigone makes no other demonstration by syllogisms and neither does he make any identification between symbolic lines of the demonstration and the premises of these syllogisms, although this may be deduced from his explanation in the preface: "The demonstration... included each one on a short line, which can be solved easily by syllogisms, because in the proposition cited as well as in that which corresponds to the citation one can find all parts of the syllogism."

Hérigone's originality resides not in demonstrating by syllogisms, but rather in recognizing that it is possible to identify all parts of the syllogism in symbolic lines, which transforms the demonstration by syllogisms into another one that is shorter and easier. Indeed, it is important to point out that Hérigone sought to introduce a new, briefer and more intelligible method for making demonstrations. Although the excess of abbreviations and new symbols may have caused his attempt to fail, there is no doubting the intelligence of Hérigone's approach.

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[39] III. SYLLOGISME. Les choses égales à une mesme, sont égales entr'elles. Mais les lignes droites AC & CB sont égales à une mesme ligne droite. Donc les lignes droites AC & BC sont égales entr'elles. /III. SYLLOGISMUS. Quae eidem aequalia sunt, inter se sunt aequalia. Sed rectae AC & BC sunt eidem rectae aequales. Igitur rectae AC & BC sunt inter se aequales. (Hérigone, 1634, I, p.2).

[40] IV. SYLLOGISME. Tout triangle qui a trois costez égaux, est equilateral. Mais le triangle ABC a trois costez égaux. Donc le triangle ABC est equilateral. /IV. SYLLOGISMUS. Omne triangulum habens tria latera aequalia, est aequilaterum. Sed triangulum ABC tria habet aequalia latera. Igitur triangulum ABC est aequilaterum. (Hérigone, 1634, I, p.2).

5.4 The usefulness of Hérigone's new method

In this section we analyze two examples in order to show how Hérigone sometimes makes improvements on Viète's examples and Clavius's *Elements*, using his new method.

5.4.1 Equations in the Algebra

The first example refers to the treatment of equations in the *Algebra*. According to Hérigone, an understanding of Euclid's *Elements* is the basis for understanding arithmetic and solving equations. In *Algebra*, Hérigone used propositions from Euclid's *Elements* to justify algebraic demonstrations. Furthermore, all instructions, procedures and rhetorical explanations for geometrical constructions are replaced by Euclid's propositions and postulates, expressed or formulated in Hérigone's symbolic language. Thus, Euclid's *Elements* are deeply entrenched in the development of Hérigone's *Algebra*.

Algebra is a section in Volume 2, which consists of 20 chapters. Hérigone accepts Viète's view that the symbols of analytic art (or algebra) can be used to represent not just numbers but also values of any abstract magnitude.[41] Indeed, Hérigone explicitly distinguishes vulgar algebra, which deals with problems expressed in terms of numbers, from specious algebra, which deals with problems expressed in more general terms, by means of species or letters. This idea is very important because it is from Viète's algebra that mathematicians began to consider objects of algebra, the letters of which represent numbers and also figures, angles and lines.

Hérigone's Notation	Modern Notation
$ab - a2\ 2\vert 2\ d2$	$xb - x^2 = d^2$
$b - a \prod d \prod a$	$\frac{(b-x)}{d} = \frac{d}{x}$
$a\ 2\vert 2\ \frac{1}{2}b + \sqrt{\begin{matrix}\{b2\frac{1}{4} \\ \{-d2\end{matrix}}$	$x = \frac{1}{2}b + \sqrt{\frac{b^2}{4} - d^2}.$

Fig. 5.7: Modern translation of notations from Hérigone's *Algebra*.

As regards the treatment of equations, Hérigone, like Viète, transforms equations into a relationship between three proportional quantities. The key is the identification of the terms of an equation, both known and unknown quantities, as terms of a proportion. However, Hérigone always specifies whether

[41] On the comparison between Viète's and Hérigone's algebra, see Massa (2008).

the required quantity is a number or a line, and in the latter case he justifies the results by using propositions from Book VI of Euclid's *Elements*. Solving equations consists of three steps: equation, proportion and solution by rule (see Figure 5.7).

He again justifies the passage between these steps with the propositions of Euclid's *Elements*. In this equation d is the mean proportional and b is the sum of the extremes, both of which are given. He emphasizes that with these data that this equation can be solved geometrically by the scholium of Euclid VI.28. In fact, in the case that b is the difference of the extremes, he claims that he uses the scholium of Euclid VI.29, and in the other cases of equations he mentions propositions in the appendix to the six first books.

Moreover, if we analyze this scholium of Euclid VI.28 (see Figure 5.8), which is used to justify the rules of the quadratic equations and compare it with those of Clavius, we can say that the statement, the presentation and the procedure are different; in other words, this scholium is not present in Clavius's *Elements*. So this scholium was added by Hérigone, as he states:

> Given the mean of three proportional [straight lines] and the addition of the extremes, find the extremes.[42]

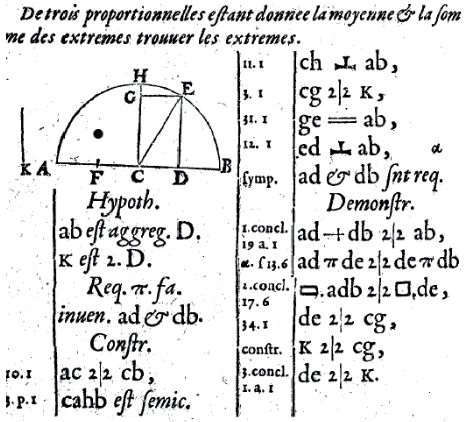

Fig. 5.8: "Scholium" of Euclid VI.28 (Hérigone, 1634, I, p.293)

The usual or classic figure in Euclid's *Elements*, with squares and rectangles or a semicircle with an inscribed triangle delimiting the diameter, is not found

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[42] De trois proportionnelles estant donnee la moyenne & la somme des extremes trouver les extremes./ E serie trium proportionalium, data media & summa extremarum invenire extremas. (Hérigone, 1634, I, p.293–294).

in this construction. We can consider this new figure as Hérigone's canonical diagram for geometric constructions for quadratic equations. Hérigone states two rules for finding the value of the unknown in an equation with three terms where the degree of the highest power is double that of the lower power, and both rules are also illustrated by this same figure. In fact, one finds this figure and the reference to this scholium throughout the *Cursus*.[43] For example, when Hérigone deals with irrational numbers in *Algebra* he again explicitly cites and uses scholium VI.28 for justifying geometrically his method for finding the root of a binomial. Using the same figure as that used in the previous scholium, in the demonstration, Hérigone states:

> To find the square root of a given binomial. Let us assume that the bigger number of the binomial is the sum of sides and the smaller number (of the binomial) is four times the rectangle determined by the sides, therefore one will find the root by the scholium 28.6, as follows.[44]

In contrast, Clavius in his *Algebra* (1608, p.150) explains three rules for finding the square root of a binomial in rhetorical language; then he gives an example for every rule with an explanation consisting of two pages. Hérigone makes the demonstration in half a page, does not use the same numerical example as Clavius, and the mathematical procedure and presentation are also very different.

This new scholium and its figure allow Hérigone to make some new demonstrations and to illustrate new rules. We show this scholium as an example of Hérigone's new threads achieved with his new method of demonstration.[45]

5.4.2 Book X Definitions

Another example focusses on Hérigone's treatment of the first definitions in Book X of Euclid's *Elements*. Book X introduces the Euclidean theory of irrationality; it is difficult and full of definitions in a geometric context, but Hérigone, unlike Clavius in his Book X, always provides examples referring to

[43] In Hérigone's *Elements* there are many examples: in propositions II.6 and II.29, in scholium of proposition II.5, in propositions VI.29, X.16, X.18 and X.19.

[44] Extraire la racine quarrée d'un binôme donné. Soit supposé que le plus grande nombre du binôme est l'aggregé des costez, & le moindre nombre le quadruple du rectangle contenu sous les costez, puis on trouvera la racine par le scholie de la 28 du 6, comme s'ensuit. /Ex dato binomio extrahere radicem quadratam. Finge maius nomen binomij dati esse aggregatum laterum, minus nomen quadruplum rectanguli sub lateribus comprehensi, deinde invenietur quaesita radix per scholium 28.6 sic (Hérigone, 1634, II, p.254).

[45] For more detailed examples and improvements see Massa (2008).

numbers and translates the demonstrations in symbolic language.[46] Moreover, he adds some scholia and a new classification to clarify these ideas.

Both Clavius and Hérigone present all eleven of Campanus's definitions; however, Hérigone makes an addition with four scholies to clarify these ideas. Definitions X.1 and X.2 define magnitudes to be commensurable when measured by the same common measure and otherwise incommensurable. Definitions X.3 and X.4 concern commensurable straight lines in square; Hérigone states that they are commensurable in square when the squares on them are measured by the same area and otherwise incommensurable in square. Let us take the first definitions: commensurability and incommensurability.

> Commensurable magnitudes are those that are measured by the same common measure.[47] But incommensurable magnitudes are those that do not have any common measure.[48]

After the fourth definition, Hérigone introduces examples in numbers. He states that the lines $a = 7$ and $b = 5$ are commensurable in length (as always, first in symbols and then with his explanation of abbreviations). The lines $b = 5$, $c = \sqrt{10}$ and $d = \sqrt{8}$ are commensurable in square because the squares 25, 10 and 8 are commensurable in length. The lines $e = \sqrt{\sqrt{10}}$ and $f = \sqrt{\sqrt{8}}$ are incommensurable in square because the squares $\sqrt{10}$ and $\sqrt{8}$ are incommensurable in length.

In definitions X.5–X.7 Hérigone describes the rational straight line. Taking a rational straight line as a reference, the other lines commensurable in length and in square are called rational. And the lines incommensurable with respect to this line are termed irrational. In definitions X.8, X.9, X.10, and X.11, Hérigone describes rational and irrational figures.

Hérigone, unlike Clavius, adds four scholia to clarify the concepts. In the first scholium, Hérigone clarifies that incommensurable magnitudes cannot become commensurable, while irrational magnitudes can become rational ones by changing the rational that one takes as a reference. In fact, for Hérigone the notions of commensurable and rational are not parallel at all.

In the scholia II and III Hérigone specifies the relation of incommensurability in numbers by taking unity as the reference. In fact, the incommensurable numbers with respect to the unit are called irrationals or "surds." However,

[46] On the treatment of Book X, there are many interesting works including Fowler (1992) and Rommevaux (2001).

[47] Commensurables grandeurs sont celles-là lesquelles sont mesurees par une mesme commune mesure. /Commensurabiles magnitudines dicuntur, quas eadem mensura metitur (Hérigone, 1634, I, p.486). The text of this statement in Latin is identical in Clavius' s *Elements*.

[48] Mais les grandeurs incommensurables sont celles-là, lesquelles n'ont aucune commune mesure. /Incommensurabiles autem sunt, quarum nullam communem mensuram contingit reperiri (Hérigone, 1634, I, p.486).

and more importantly, he specifies the idea of a rational line expressed by an irrational number that can be rational if its square is expressed by a rational number.

In scholium IV Hérigone, unlike Clavius, introduces a "new" classification of the rational lines commensurable in length to each other according to three types illustrated by a geometric construction (see figure 5.9).

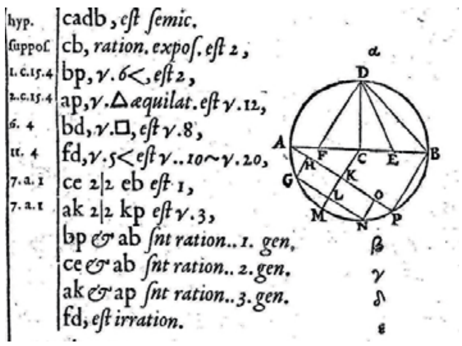

Fig. 5.9: Classification of the rational lines in Book X (Hérigone, 1634, I, p.491).

Having constructed an example, he takes the semi-diameter of the value 2 as the unit of measure; he then constructs the other lines using as major premises both corollaries of Euclid IV.15, which states: "In a given circle to inscribe an equilateral and equiangular hexagon,"[49] Euclid IV.6 states: "In a given circle to inscribe a square."[50], Euclid IV.11 states: "In a given circle to inscribe an equilateral and equiangular pentagon,"[51] and the common notion Euclid I. axiom. 7 states: "And things which are half of another thing or equal things are also all equal to each other."[52] He concludes that if $cb = 2$ is the rational line of reference; $bp = 2$ and $ab = 4$ are rational lines of the first type, because they are commensurable in length and one of them is equal to the rational reference; $ce = 1$ and $ab = 4$ are rational lines of the second type because they are commensurable in length and none of them is equal to the

[49] En un cercle donné, inscrire un hexagone equilateral & equiangle. /In dato circulo, hexagonum & equilaterum & equiangulum inscribere (Hérigone, 1634, I, p.169).

[50] Dans un cercle donné, inscrire un carré. /In dato circulo quadratum describere (Hérigone, 1634, I, p.156).

[51] En un cercle donné, inscrire un pentagone equilateral & equiangle. /In dato circulo, pentagonum & equilaterum & equiangulum inscribere (Hérigone, 1634, I, p.161).

[52] Et les choses qui sont moitiés d'une mesme, ou des choses egales, sont aussi egales entre elles. /Et quae eiusdem, vel aequalium sunt dimidia inter se sunt aequalia (Hérigone, 1634, I, p. c.iiij).

rational reference. For the third type $ak = \sqrt{3}$ and $ap = \sqrt{12}$ are rational because they are only commensurable in square with the rational reference and $fd = \sqrt{10 - \sqrt{20}}$ is irrational because it is neither commensurable in square nor in length.

Hérigone later uses this new classification of the rational lines and this figure to solve a problem with irrational numbers in the *Algebra*. The question consists in finding the side of a regular pent decagon inscribed in a circle. Hérigone states: "To find the side of a regular pent decagon inscribed in a given circle."[53]

5.5 Some final remarks

The first remark to be made is that, since Hérigone mentioned that he used the Latin version of Clavius (1589) to write his *Elements*, we have verified the statements of definitions in Latin and they turn out to be mostly identical. However, unlike Clavius, after every statement Hérigone gives no rhetorical explanations. Moreover, Hérigone adds some scholia and an appendix in order to explain the mathematics better.

He reformulates Clavius's *Elements* by using his symbolic language in an original way. Thus, Hérigone in the *Cursus* avoids rhetorical explanations and seeks to express all phrases symbolically. The steps are justified by citations referring to the propositions, axioms, postulates and definitions from Euclid's *Elements*, which are formulated in symbolic language in Volume 1 as well. We may surmise that Hérigone's presentation of this justification is once more a reflection of the great significance that Euclid's *Elements* held for him.

After showing Hérigone's examples in our Section 3, his procedure of replacing the rhetorical language of Euclid's *Elements* by symbolic language to make demonstrations in the *Cursus* becomes clear. This new method of demonstration using a universal language and logical sentences through an axiomatic-deductive reasoning is absolutely original and offers us the logical and clear structure of his thinking. Moreover, Hérigone emphasizes that his method is useful for both pure and mixed mathematics and he applies it in all parts of the *Cursus*. Perhaps the idea of extending his method to all mathematics arose from his reading of Clavius's *Elements*. Indeed, Clavius in the first demonstration of the first book after the demonstration by syllogisms claims that in this manner one can solve all Euclid's propositions as well as those of all other mathematicians.

[53] Quaestion III. Cap. XIX. Trouver le coté d'un quindecagonregulier inscrit dans un cercle donné. /Invenire latus quintidecagoni ordinati dato circuli inscripti (Hérigone, 1634, II, p.261).

However, in order to highlight the highly unusual relationship between the classical mathematics that Euclid's work represents and new algebra as it appears in the work of Hérigone, we would like to discuss whether the symbolic language introduced in Hérigone's *Elements* is useful to obtain new results or whether it is a different way of arriving at the same results. In other words, did Hérigone actually perform an algebraization of the *Elements*? I believe that he did not, at least not completely; Hérigone indeed translates the different notions, interprets the statement and the demonstrations in terms of symbolic notation, and at the same time gives numerical examples, but without overlooking the geometric context. He uses figures, abbreviations and symbols to establish and reinforce these meanings as well as to make them meaningful for his readers.

The strategies employed by Hérigone in order to render Euclid's geometry and all his mathematics intelligible to his audience also enable him to make some improvements. In my previous article on Hérigone's *Algebra* and in the examples from Section 4, it was shown that this different writing in logical statements allows him to obtain some new demonstrations, some new rules, some new paths and some new classifications. In fact, in the preface "to the reader," in the first volume of the *Cursus*, Hérigone lays claim to his contributions by stating:

> Those who love these divine sciences [Mathematics] may judge what I have contributed on my own behalf in every part of this *Cursus*, which I trust will prove to be of use and profit to them.[54]

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[54] Ceux qui aiment ces divines sciences iugeront ce que j'ay apporté du mien en chacune partie de ce Cours, que je souhaite qu'il leur soit utile & profitable. /Quid autem in singulis huius Cursus partibus praestiterim, iudicabunt studiosi, quibus opto hunc meum laborem utilem esse (Hérigone, 1634, I, *Ad Lectorem*).

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